Joint functional laws for the Wiener process, its local time and principal value

by

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Summary. We present joint functional iterated logarithm laws for the Wiener process and its local time, as well as for the Wiener process and the principal value of its local times.

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1 Introduction

Let $\{W(t);\ t \geqslant 0\}$ be a one-dimensional Wiener process with $W(0) = 0$, and let $\{L(t, x);\ t \geqslant 0,\ x \in \mathbb{R}\}$ denote its local time process, jointly continuous in t and x. For any Borel function $f \geqslant 0$,

$$\int_0^t f(W(s))\,\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L(t,x)\,\mathrm{d}x, \qquad t \geqslant 0.$$

Put $L(t, 0) = L(t)$ and

$$\begin{aligned} U_t(x) &:= \frac{W(xt)}{\sqrt{2t\log\log t}}, \\ V_t(x) &:= \frac{L(xt)}{\sqrt{2t\log\log t}}, \qquad x \in [0,1]. \end{aligned}$$

We consider $x \mapsto U_t(x)$ and $x \mapsto V_t(x)$ as elements of the space $\mathcal{C} = \mathcal{C}[0, 1]$ of continuous functions with metric

$$d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|.$$

Recall the celebrated functional law of the iterated logarithm (FLIL) for W due to Strassen [18]:

Theorem A *With probability one, the set* $\{U_t\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}$*, with limit set equal to*

$$\mathcal{S} := \left\{ f \in \mathcal{C} :\ f(0) = 0,\ f \text{ is absolutely continuous, with } \int_0^1 (f'(x))^2\,\mathrm{d}x \leqslant 1 \right\}.$$

Using that $\{L(t),\ t \geqslant 0\}$ has the same distribution as $\{\sup_{s\in[0,t]} W(s),\ t \geqslant 0\}$, one can easily obtain (cf. Csáki and Révész [10], Mueller [16], Chen [4]),

Theorem B *With probability one, the set* $\{V_t\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}$*, with limit set equal to*

$$\mathcal{S}_M := \{g \in \mathcal{S} :\ g \text{ is non-decreasing}\}.$$

Our concern in this paper is to prove a joint FLIL for the vector $\{(U_t(x), V_t(x)),\ x \in [0, 1]\}_{t\geqslant 1}$. We are working on the space $\mathcal{C}^{(2)} := \mathcal{C} \times \mathcal{C}$ with metric

$$d((f_1, g_1), (f_2, g_2)) = \sup_{x\in[0,1]} \sqrt{(f_1(x) - f_2(x))^2 + (g_1(x) - g_2(x))^2}.$$

Our first main result is

Theorem 1.1 *With probability one, the set* $\{(U_t, V_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\mathcal{S}_J^{(2)} := \left\{ (f,g) : f \in \mathcal{S}, g \in \mathcal{S}_M, \int_0^1 (f'(x))^2 + (g'(x))^2)\, \mathrm{d}x \leqslant 1,\ f(x)g'(x) = 0 \text{ a.e.} \right\}.$$

Remark 1.2 If W and L were independent, then the limit set would be equal to (cf. for example [9])

$$\mathcal{S}^{(2)} = \left\{ (f,g) : f \in \mathcal{S}, g \in \mathcal{S}_M, \int_0^1 (f'(x))^2 + (g'(x))^2)\, \mathrm{d}x \leqslant 1 \right\}.$$

Obviously, L being the local time of W is not independent of W. The extra condition $f(x)g'(x) = 0$ a.e. expresses the fact that if W is away from zero, then L does not increase. □

We are also interested in studying the process

$$Y(t) = \int_0^t \frac{\mathrm{d}s}{W(s)}, \qquad t \geqslant 0.$$

Rigorously speaking, the integral $\int_0^t \mathrm{d}s/W(s)$ should be considered in the sense of Cauchy's principal value, i.e., $Y(t)$ is defined by

$$Y(t) = \lim_{\varepsilon \to 0+} \int_0^t \frac{\mathrm{d}s}{W(s)} \mathbb{1}_{\{|W(s)| \geqslant \varepsilon\}} = \int_0^\infty \frac{L(t,x) - L(t,-x)}{x} \mathrm{d}x. \tag{1.1}$$

Since $x \mapsto L(t,x)$ is Hölder continuous of order ν, for any $\nu < 1/2$, the integral on the right hand side of (1.1) is well-defined.

The study of Cauchy's principal value of Brownian local time goes back at least to Itô and McKean [15], and has become very active since the late 70s, due to applications in various branches of stochastic analysis. For a detailed account of various motivations, historical facts and general properties of principal values of local times, we refer to the recent collection of research papers in Yor [20], to Chapter 10 of the lecture notes by Yor [21], and to the survey paper by Yamada [19].

The process $Y(\cdot)$ defined in (1.1) is continuous, having zero quadratic variation. It is easily seen that $Y(\cdot)$ inherits a scaling property from Brownian motion, namely, for any fixed $a > 0$, $t \mapsto a^{-1/2} Y(at)$ has the same law as $t \mapsto Y(t)$. Although the aforementioned zero quadratic variation property distinguishes $Y(\cdot)$ from Brownian motion (in particular, $Y(\cdot)$ is not a semimartingale), it is a kind of folklore that Y behaves somewhat like a Brownian motion. Hu and Shi [14] proved a law of the iterated logarithm for $Y(\cdot)$:

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{8t\log\log t}} = 1 \qquad \text{a.s.}$$

FLIL for Y was not known before. Here we show that similarly to Theorem 1.1, a joint FLIL for W and Y holds. Introduce

$$Z_t(x) = \frac{Y(xt)}{\sqrt{8t\log\log t}}, \qquad 0 \leqslant x \leqslant 1.$$

Our next result is

Theorem 1.3 *With probability one the set $\{(U_t, Z_t)\}_{t\geqslant 1}$ is relatively compact in $\mathcal{C}^{(2)}$, with limit set equal to*

$$\widetilde{\mathcal{S}}_J^{(2)} = \left\{(f,g)\,:\, f\in\mathcal{S}, g\in\mathcal{S}, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \leqslant 1,\; f(x)g'(x) = 0 \text{ a.e.}\right\}.$$

Some consequences are as follows.

Corollary 1.4 *With probability one, the set $\{Z_t\}_{t\geqslant 1}$ is relatively compact in $\mathcal{C}$, with limit set equal to $\mathcal{S}$ given in Theorem A.*

Corollary 1.5 *With probability one, the set $\{(U_t(1), V_t(1))\}_{t\geqslant 1}$ is relatively compact in $\mathbb{R}^2$ with limit set equal to*

$$\{(x,y)\in\mathbb{R}^2 \,:\, y\geqslant 0,\; |x|+y\leqslant 1\}.$$

Corollary 1.6 *With probability one, the set $\{(U_t(1), Z_t(1))\}_{t\geqslant 1}$ is relatively compact in $\mathbb{R}^2$ with limit set equal to*

$$\{(x,y)\in\mathbb{R}^2 \,:\, |x|+|y|\leqslant 1\}.$$

The organization of the paper is as follows: In Section 2 we present some preliminary results for the distribution of the Wiener process, local time and principal value, as well as certain estimates for the increments of the processes concerned. In Section 3 we prove Theorem 1.1, while Section 4 is devoted to the proof of Theorem 1.3. In Section 5 we prove the Corollaries. Some further remarks and consequences are given in Section 6.

Throughout the paper, for any $x\in\mathbb{R}$, we denote $\mathbb{P}^x$ the probability under which the Wiener process W starts from $W(0)=x$ (thus $\mathbb{P}=\mathbb{P}^0$); unimportant constants (which are finite and positive) are denoted by the letter c with subscript.

2 Preliminaries

2.1 Distribution results for Wiener process and its local time

The proof of Theorem 1.1 is based on the formula 1.3.8 in Borodin and Salminen [3]:

$$\mathbb{P}^{z_1}(L(t) \in \mathrm{d}y,\, W(t) \in \mathrm{d}z_2) = p_t(y; z_1, z_2)\,\mathrm{d}y\,\mathrm{d}z_2, \qquad y > 0, \tag{2.1}$$

and

$$\mathbb{P}^{z_1}(L(t) = 0,\, W(t) \in \mathrm{d}z_2) = q_t(z_1, z_2)\,\mathrm{d}z_2, \tag{2.2}$$

where

$$p_t(y; z_1, z_2) = \frac{y + |z_1| + |z_2|}{t}\varphi_t(y + |z_1| + |z_2|), \tag{2.3}$$

$$q_t(z_1, z_2) = \varphi_t(z_2 - z_1) - \varphi_t(|z_1| + |z_2|), \tag{2.4}$$

$$\varphi_t(x) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{x^2}{2t}\right). \tag{2.5}$$

2.2 Distribution results for Wiener process and principal value

First recall some results for principal value. Biane and Yor [1] proved the following result: Let $\{B(s),\, 0 \leqslant s \leqslant 1\}$ be a Brownian bridge, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{B(s)} < x\right) &= \frac{|x|}{2}\sum_{n=1}^{\infty}\exp\left(-\frac{(2n-1)^2x^2}{8}\right)\\ &\geqslant \frac{|x|}{2}\exp\left(-\frac{x^2}{8}\right).\end{aligned} \tag{2.6}$$

It follows that for $0 < \alpha < \beta$

$$\mathbb{P}\left(\int_0^1 \frac{\mathrm{d}s}{B(s)} \in (\alpha, \beta)\right) \geqslant 2\left(\exp\left(-\frac{\alpha^2}{8}\right) - \exp\left(-\frac{\beta^2}{8}\right)\right). \tag{2.7}$$

It was proved in [7] (cf. (2.11), (2.14) and (2.16) there) that for any $\delta > 0$ there exists $c_1(\delta) > 0$ such that for all $s > 0$ and $x > 0$,

$$\sup_{z\in\mathbb{R}} \mathbb{P}^z(|Y(s)| > x) \leqslant c_1(\delta)\exp\left(-\frac{x^2}{(8+\delta)s}\right). \tag{2.8}$$

Lemma 2.1 *Let $s > 0$, $\lambda > 0$, $\delta > 0$ and $0 < \varepsilon < 1$. For $(a, \alpha, z) \in \mathbb{R}^3$, define*

$$I = I(a, \alpha, z) := \mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\ \ \alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda\right). \tag{2.9}$$

Then

$$I \leqslant \frac{\lambda}{\sqrt{s}} \exp\left(-\frac{(|a-z| - 2\varepsilon\lambda)^2 - 4\varepsilon^2\lambda^2}{2s}\right). \tag{2.10}$$

Moreover, if $|\alpha| \geqslant 4\varepsilon\lambda$, then

$$I \leqslant c_1(\delta) \exp\left(-\frac{(|\alpha| - 4\varepsilon\lambda)^2}{(8+\delta)s}\right), \tag{2.11}$$

where $c_1(\delta)$ is the constant in (2.8).

Proof: Observe that

$$I \leqslant \mathbb{P}^z(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda) = \mathbb{P}\left(\frac{a-z}{\sqrt{s}} \leqslant N \leqslant \frac{a - z + 2\varepsilon\lambda}{\sqrt{s}}\right),$$

where N is a standard normal variable. Hence (2.10) follows from a straightforward Gaussian estimate.

Now for $|\alpha| \geqslant 4\varepsilon\lambda$, we have

$$I \leqslant \mathbb{P}^z(\alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda) \leqslant \mathbb{P}^z(|Y(s)| \geqslant |\alpha| - 4\varepsilon\lambda),$$

which implies (2.11) by means of (2.8). □

For the lower estimates we prove several lemmas.

Lemma 2.2 *For $\alpha > 0$, $\beta - \alpha > 4$, $0 < \delta < 1$ we have*

$$\mathbb{P}(|W(1)| \leqslant 1,\ \alpha \leqslant Y(1) \leqslant \beta) \geqslant c_2(\delta) \exp\left(-\frac{(\alpha+1)^2}{8(1-\delta)}\right), \tag{2.12}$$

where $c_2(\delta)$ is a constant depending only on δ.

Proof: Let

$$\begin{aligned} G &:= \sup\{t :\ t \leqslant 1,\ W(t) = 0\}, \\ B(s) &:= \frac{W(sG)}{\sqrt{G}}, \qquad s \in [0,1]. \end{aligned}$$

It is known that $(B(s),\ s \in [0,1])$, G and $(\frac{W(G+s(1-G))}{\sqrt{1-G}},\ s \in [0,1])$ are independent, and that $(B(s),\ s \in [0,1])$ is a (standard) Brownian bridge.

We have

$$\begin{aligned}
& \mathbb{P}(|W(1)| \leqslant 1,\, \alpha \leqslant Y(1) \leqslant \beta) \\
\geqslant\ & \mathbb{P}(|W(1)| \leqslant 1,\, \alpha+1 \leqslant Y(G) \leqslant \beta - 1,\, |Y(1) - Y(G)| \leqslant 1,\, G \geqslant 1-\delta) \\
=\ & \int_{1-\delta}^{1} \mathbb{P}(|W(1)| \leqslant 1,\, \alpha+1 \leqslant Y(\kappa) \leqslant \beta - 1,\, |Y(1) - Y(\kappa)| \leqslant 1 \,|\, G=\kappa)\, \mathbb{P}(G \in \mathrm{d}\kappa) \\
=\ & \int_{1-\delta}^{1} \mathbb{P}(\alpha+1 \leqslant Y(\kappa) \leqslant \beta - 1 \,|\, G=\kappa) \times \\
& \qquad\qquad \times \mathbb{P}(|W(1)| \leqslant 1,\, |Y(1) - Y(\kappa)| \leqslant 1 \,|\, G=\kappa)\, \mathbb{P}(G \in \mathrm{d}\kappa).
\end{aligned}$$

Since under the condition $G=\kappa$, $Y(\kappa)/\sqrt{\kappa}$ has the same distribution as $\int_0^1 \mathrm{d}s/B(s)$, where B is a Brownian bridge, we get from (2.7)

$$\begin{aligned}
\mathbb{P}(\alpha+1 \leqslant Y(\kappa) \leqslant \beta - 1 \,|\, G=\kappa) &\geqslant 2\left(\exp\left(-\frac{(\alpha+1)^2}{8\kappa}\right) - \exp\left(-\frac{(\beta-1)^2}{8\kappa}\right)\right) \\
&\geqslant 2(1-\mathrm{e}^{-1})\exp\left(-\frac{(\alpha+1)^2}{8\kappa}\right) \\
&\geqslant 2(1-\mathrm{e}^{-1})\exp\left(-\frac{(\alpha+1)^2}{8(1-\delta)}\right).
\end{aligned}$$

This gives (2.12), with

$$c_2(\delta) := 2(1-\mathrm{e}^{-1})\mathbb{P}(|W(1)| \leqslant 1,\, |Y(1)-Y(G)| \leqslant 1,\, G \geqslant 1-\delta).$$

The lemma is proved. □

Now we introduce the notation

$$T_b := \inf\{t:\ t \geqslant 0,\, W(t) = b\}. \tag{2.13}$$

By the reflection principle, we have for all $u > 0$ and $(a,z) \in \mathbb{R}^2$,

$$\mathbb{P}^z(T_a \leqslant u) = 2\overline{\Phi}\left(\frac{|z-a|}{\sqrt{u}}\right), \tag{2.14}$$

where $\overline{\Phi}(x) := \mathbb{P}(N > x)$ is the standard Gaussian tail distribution function.

In the sequel we shall use the inequalities:

$$\overline{\Phi}(x) \geqslant \frac{1}{(2\pi)^{1/2}}\left(\frac{1}{x} - \frac{1}{x^3}\right)\exp\left(-\frac{x^2}{2}\right), \qquad x \geqslant 1, \tag{2.15}$$

$$\overline{\Phi}(x) \leqslant \frac{1}{2}\exp\left(-\frac{x^2}{2}\right), \qquad x > 0. \tag{2.16}$$

(For (2.16), see Proposition II.1.8 of Revuz and Yor [17].)

Lemma 2.3 *For $s > 0$, $0 < \delta < 1$, $z \in \mathbb{R}$ we have*

$$\mathbb{P}^z(T_0 \leqslant \delta s,\ |Y(T_0)| \leqslant 2\sqrt{s}) \geqslant c_3(\delta)\,\overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}}\right). \tag{2.17}$$

Proof: By symmetry, it suffices to prove (2.17) for $z > 0$ (there is nothing to prove if $z = 0$). Assuming first $z > \sqrt{s}$, we have

$$\begin{aligned} & \mathbb{P}^z(T_0 \leqslant \delta s,\ |Y(T_0)| \leqslant 2\sqrt{s}\,) \\ \geqslant\ & \mathbb{P}^z(T_0 - T_{\sqrt{s}} \leqslant \delta(1-\delta)s,\ T_{\sqrt{s}} \leqslant \delta^2 s,\ Y(T_0) - Y(T_{\sqrt{s}}) \leqslant \sqrt{s}) \\ =\ & \mathbb{P}^{\sqrt{s}}(T_0 \leqslant \delta(1-\delta)s, Y(T_0) \leqslant \sqrt{s}\,)\,\mathbb{P}^z(T_{\sqrt{s}} \leqslant \delta^2 s), \end{aligned}$$

where we used the fact that $T_{\sqrt{s}} \leqslant \delta^2 s$ implies $Y(T_{\sqrt{s}}) \leqslant T_{\sqrt{s}}/\sqrt{s} \leqslant \delta^2\sqrt{s} < \sqrt{s}$.

By scaling, $\mathbb{P}^{\sqrt{s}}(T_0 \leqslant \delta(1-\delta)s, Y(T_0) \leqslant \sqrt{s}\,)$ is a positive constant depending only on δ. In view of (2.14), we have proved (2.17) in case $z > \sqrt{s}$.

If $0 < z \leqslant \sqrt{s}$, we have, by scaling,

$$\begin{aligned} \mathbb{P}^z(T_0 \leqslant \delta s,\ |Y(T_0)| \leqslant 2\sqrt{s}\,) &= \mathbb{P}^1\left(T_0 \leqslant \frac{\delta s}{z^2},\ |Y(T_0)| \leqslant \frac{2\sqrt{s}}{z}\right) \\ &\geqslant \mathbb{P}^1\left(T_0 \leqslant \delta,\ |Y(T_0)| \leqslant 2\right) \\ &=: c_4(\delta), \end{aligned}$$

from which (2.17) follows. □

Lemma 2.4 *Let $s > 0$, $\varepsilon > 0$, $\lambda > 0$, $0 < \delta < 1$, $(\alpha, z) \in \mathbb{R}^2$ be such that $\varepsilon\lambda > 8\sqrt{s}$. Then we have*

$$\begin{aligned} & \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\ \alpha \leqslant Y(s) \leqslant \alpha + 4\varepsilon\lambda) \\ \geqslant\ & c_5(\delta)\exp\left(-\frac{(|\alpha| + 2\varepsilon\lambda)^2}{8s(1-\delta)^2}\right)\overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}}\right). \end{aligned} \tag{2.18}$$

Proof: Define, for $n \geqslant 1$,

$$I_{\lambda,z}(\alpha, n) := \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\ \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda).$$

Note that $I_{\lambda,z}(\alpha, n)$ is non-decreasing in n. Moreover,

$$\begin{aligned} I_{\lambda,z}(\alpha,n) &\geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda,\, T_0 \leqslant \delta s) \\ &= \int_0^{\delta s} \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha + n\varepsilon\lambda \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) \\ &\geqslant \int_0^{\delta s} \mathbb{P}^z(A_\tau \cap B_\tau(n) \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau), \end{aligned}$$

where

$$\begin{aligned} A_\tau &:= \{|Y(\tau)| \leqslant 2\sqrt{s}\}, \\ B_\tau(n) &:= \{|W(s)| \leqslant \varepsilon\lambda,\, \alpha + 2\sqrt{s} \leqslant Y(s) - Y(\tau) \leqslant \alpha + n\varepsilon\lambda - 2\sqrt{s}\}. \end{aligned}$$

Under the condition $\{W(0) = z,\, T_0 = \tau\}$, A_τ and $B_\tau(n)$ are independent, so that

$$I_{\lambda,z}(\alpha,n) \geqslant \int_0^{\delta s} \mathbb{P}^z(A_\tau \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) \times \inf_{\tau \in (0,\,\delta s)} \mathbb{P}^z(B_\tau(n) \,|\, T_0 = \tau).$$

By Lemma 2.3,

$$\begin{aligned} \int_0^{\delta s} \mathbb{P}^z(A_\tau \,|\, T_0 = \tau)\, \mathbb{P}^z(T_0 \in \mathrm{d}\tau) &= \mathbb{P}^z(|Y(T_0)| \leqslant 2\sqrt{s},\, T_0 \leqslant \delta s) \\ &\geqslant c_3(\delta)\, \overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}} \right), \end{aligned}$$

whereas according to Lemma 2.2, and by scaling,

$$\begin{aligned} &\mathbb{P}^z(B_\tau(1) \,|\, T_0 = \tau) \\ &= \mathbb{P}(|W(s-\tau)| \leqslant \varepsilon\lambda,\, \alpha + 2\sqrt{s} \leqslant Y(s-\tau) \leqslant \alpha + \varepsilon\lambda - 2\sqrt{s}) \\ &\geqslant \mathbb{P}\left(|W(1)| \leqslant 1,\, \frac{\alpha + 2\sqrt{s}}{\sqrt{s-\tau}} \leqslant Y(1) \leqslant \frac{\alpha + \varepsilon\lambda - 2\sqrt{s}}{\sqrt{s-\tau}} \right). \end{aligned}$$

Assume $\alpha \geqslant 0$ for the moment. By Lemma 2.2,

$$\mathbb{P}^z(B_\tau(1) \,|\, T_0 = \tau) \geqslant c_2(\delta)\, \exp\left(-\frac{(\alpha + \varepsilon\lambda)^2}{8s(1-\delta)^2} \right),$$

which yields

$$I_{\lambda,z}(\alpha,1) \geqslant c_6(\delta)\, \exp\left(-\frac{(\alpha+\varepsilon\lambda)^2}{8s(1-\delta)^2} \right) \overline{\Phi}\left(\frac{||z| - \sqrt{s}|}{\delta\sqrt{s}} \right), \qquad \alpha \geqslant 0, \tag{2.19}$$

with $c_6(\delta) := c_3(\delta)c_2(\delta)$. Since $I_{\lambda,z}(\alpha, 4) \geqslant I_{\lambda,z}(\alpha, 1)$, this yields (2.18) in case $\alpha \geqslant 0$.

To treat the case $\alpha \leqslant -\varepsilon\lambda$, we observe that

$$\begin{aligned} I_{\lambda,z}(\alpha,4) &\geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha \leqslant Y(s) \leqslant \alpha+\varepsilon\lambda) \\ &= \mathbb{P}^{-z}(|W(s)| \leqslant \varepsilon\lambda,\, -\alpha-\varepsilon\lambda \leqslant Y(s) \leqslant -\alpha), \end{aligned}$$

the last identity following via replacing W by $-W$. This gives $I_{\lambda,z}(\alpha,4) \geqslant I_{\lambda,-z}(-\alpha-\varepsilon\lambda,1)$. Since $-\alpha-\varepsilon\lambda \geqslant 0$, we are entitled to apply (2.19) to deduce (2.18).

It remains to study the situation $\alpha \in (-\varepsilon\lambda, 0)$. In this case,

$$I_{\lambda,z}(\alpha,4) \geqslant \mathbb{P}^z(|W(s)| \leqslant \varepsilon\lambda,\, \alpha+\varepsilon\lambda \leqslant Y(s) \leqslant \alpha+2\varepsilon\lambda) = I_{\lambda,z}(\alpha+\varepsilon\lambda,1),$$

which yields (2.18) in view of (2.19).

Lemma 2.4 is proved. □

Lemma 2.5 *For $s>0$, $\varepsilon>0$, $\lambda>0$, $(a,z)\in\mathbb{R}^2$ such that $\varepsilon^2\lambda^2 \geqslant 2s$, $az>0$, and*

$$|z| > \frac{s}{2\varepsilon\lambda} + 3\varepsilon\lambda, \quad |a| > \frac{s}{2\varepsilon\lambda} + 3\varepsilon\lambda$$

we have

$$\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda,\, |Y(s)| \leqslant 2\varepsilon\lambda) \geqslant \frac{1}{2}\exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right). \tag{2.20}$$

Proof: It suffices to prove the lemma for $z > \frac{s}{2\varepsilon\lambda} + \varepsilon\lambda$ and $a > \frac{s}{2\varepsilon\lambda}+\varepsilon\lambda$ (then by symmetry, it will also cover the case $a<0$ and $z<0$). We have,

$$\begin{aligned} &\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda,\, |Y(s)| \leqslant 2\varepsilon\lambda) \\ \geqslant\ &\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u) > \frac{s}{2\varepsilon\lambda},\, a \leqslant W(s) \leqslant a+2\varepsilon\lambda\right) \\ =\ &\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda) \\ &-\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u) \leqslant \frac{s}{2\varepsilon\lambda},\, a \leqslant W(s) \leqslant a+2\varepsilon\lambda\right). \end{aligned}$$

By the reflection principle,

$$\begin{aligned} &\mathbb{P}^z\left(\inf_{0\leqslant u\leqslant s} W(u) \leqslant \frac{s}{2\varepsilon\lambda},\, a \leqslant W(s) \leqslant a+2\varepsilon\lambda\right) \\ =\ &\mathbb{P}^z\left(\frac{s}{\varepsilon\lambda} - a - 2\varepsilon\lambda \leqslant W(s) \leqslant \frac{s}{\varepsilon\lambda} - a\right) \\ \leqslant\ &\mathbb{P}\left(\frac{W(s)}{\sqrt{s}} \leqslant -\frac{a+z-\frac{s}{\varepsilon\lambda}}{\sqrt{s}}\right) \end{aligned}$$

$$\leqslant \frac{1}{2}\exp\left(-\frac{\left(a+z-\frac{s}{\varepsilon\lambda}\right)^2}{2s}\right),$$

the last inequality following from (2.16). On the other hand,

$$\begin{aligned}\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda) &= \mathbb{P}\left(\frac{a-z}{\sqrt{s}} \leqslant \frac{W(s)}{\sqrt{s}} \leqslant \frac{a-z+2\varepsilon\lambda}{\sqrt{s}}\right) \\ &\geqslant \frac{2\varepsilon\lambda}{\sqrt{2\pi s}}\exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right) \\ &\geqslant \exp\left(-\frac{(|a-z|+2\varepsilon\lambda)^2}{2s}\right).\end{aligned}$$

Since $a+z-s/(\varepsilon\lambda) \geqslant |a-z|+2\varepsilon\lambda$, we obtain (2.20). □

Lemma 2.6 *For $s>0$, $\varepsilon>0$, $\lambda>0$, $(a,z)\in\mathbb{R}^2$ such that $az<0$, $|a|>2\varepsilon\lambda+\sqrt{s}$ and $\min(\varepsilon\lambda/2,\,|z|)>\sqrt{s}$, we have*

$$\mathbb{P}^z(a \leqslant W(s) \leqslant a+2\varepsilon\lambda,\ |Y(s)| \leqslant 2\varepsilon\lambda) \geqslant c_7(\delta)\,\overline{\Phi}\left(\frac{|a-z|+2\varepsilon\lambda}{(1-\delta)\sqrt{s}}\right). \tag{2.21}$$

Proof: First we show for $a>\sqrt{u}$, $\varepsilon\lambda>2\sqrt{u}$,

$$P(u) := \mathbb{P}(a \leqslant W(u) \leqslant a+2\varepsilon\lambda,\ |Y(u)| \leqslant 2\sqrt{u}) \geqslant c_8(\delta)\ \exp\left(-\frac{a^2}{2(1-\delta)u}\right). \tag{2.22}$$

Define $G_{\sqrt{u}} := \sup\{t \leqslant u :\ W(t)=\sqrt{u}\}$. Then

$$\begin{aligned}P(u) &\geqslant \int_0^{\delta u} \mathbb{P}(a \leqslant W(u) \leqslant a+2\varepsilon\lambda,\ |Y(u)| \leqslant 2\sqrt{u}\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(G_{\sqrt{u}}\in \mathrm{d}v) \\ &\geqslant \int_0^{\delta u} \mathbb{P}(|Y(v)| \leqslant \sqrt{u}\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(a \leqslant W(u) \leqslant a+2\varepsilon\lambda\,|\,G_{\sqrt{u}}=v)\,\mathbb{P}(G_{\sqrt{u}}\in \mathrm{d}v).\end{aligned}$$

Under the condition $\{G_{\sqrt{u}}=v\}$, $\{M(r) := \frac{W(v+r(u-v))-\sqrt{u}}{\sqrt{u-v}},\ r\in[0,1]\}$ is a standard Brownian meander, and from the well-known identity (Biane and Yor [1]) $\mathbb{P}(M(1) \leqslant x) = 1-\exp(-x^2/2)$, we get that, for $v\in[0,\delta u]$, $a>\sqrt{u}$ and $\varepsilon\lambda>2\sqrt{u}$,

$$\begin{aligned}\mathbb{P}(a \leqslant W(u) \leqslant a+2\varepsilon\lambda\,|\,G_{\sqrt{u}}=v) &= \mathbb{P}\left(\frac{a-\sqrt{u}}{\sqrt{u-v}} \leqslant M(1) \leqslant \frac{a-\sqrt{u}+2\varepsilon\lambda}{\sqrt{u-v}}\right) \\ &= \exp\left(-\frac{(a-\sqrt{u})^2}{2(u-v)}\right) - \exp\left(-\frac{(a-\sqrt{u}+2\varepsilon\lambda)^2}{2(u-v)}\right) \\ &\geqslant c_9\ \exp\left(-\frac{(a-\sqrt{u})^2}{2(u-v)}\right)\end{aligned}$$

$$\geqslant \; c_9 \exp\left(-\frac{a^2}{2(1-\delta)u}\right),$$

where $c_9 > 0$ is an absolute constant. Hence

$$P(u) \geqslant c_{10}(\delta)\, \exp\left(-\frac{a^2}{2(1-\delta)u}\right),$$

with

$$\begin{aligned} c_{10}(\delta) &:= c_9 \int_0^{\delta u} \mathbb{P}(|Y(v)| \leqslant \sqrt{u} \,|\, G_{\sqrt{u}} = v)\, \mathbb{P}(G_{\sqrt{u}} \in \mathrm{d}v) \\ &= c_9\, \mathbb{P}(G_{\sqrt{u}} \leqslant \delta u,\, |Y(G_{\sqrt{u}})| \leqslant \sqrt{u}\,), \end{aligned}$$

which, by scaling, does not depend on u. This yields (2.22).

We now start proving (2.21). Let $\varepsilon\lambda > 2\sqrt{s}$. Let T_0 and $T_{-\sqrt{s}}$ be as in (2.13). It suffices to prove (2.21) for $z < -\sqrt{s}$ and $a > \sqrt{s}$ (then by symmetry, it will also cover the case $z > \sqrt{s}$, $a < -2\varepsilon\lambda - \sqrt{s}\,$). Since $|Y(T_{-\sqrt{s}})| \leqslant \sqrt{s}$ under $\mathbb{P}^z$ (recalling that $z < -\sqrt{s}$), we have

$$\begin{aligned} &\mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\, |Y(s)| \leqslant 2\varepsilon\lambda\right) \\ \geqslant\; &\mathbb{P}^z\big(T_0 - T_{-\sqrt{s}} \leqslant \delta s,\, |Y(T_0) - Y(T_{-\sqrt{s}})| \leqslant \sqrt{s}, \\ &\qquad a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\, |Y(s) - Y(T_{-\sqrt{s}})| \leqslant 2\sqrt{s}\big). \end{aligned}$$

By the strong Markov property at times $T_{-\sqrt{s}}$ and T_0, we get:

$$\begin{aligned} &\mathbb{P}^z\left(a \leqslant W(s) \leqslant a + 2\varepsilon\lambda,\, |Y(s)| \leqslant 2\varepsilon\lambda\right) \\ \geqslant\; &\int_0^{\delta s} \left(\int_0^{s-y} P(s-h-y)\, \mathbb{P}^z(T_{-\sqrt{s}} \in \mathrm{d}h)\right) \mathbb{P}^{-\sqrt{s}}\big(T_0 \in \mathrm{d}y,\, |Y(T_0)| \leqslant \sqrt{s}\big). \end{aligned} \tag{2.23}$$

By (2.22),

$$\begin{aligned} P(s-h-y) &\geqslant c_8(\delta)\, \exp\left(-\frac{a^2}{2(1-\delta)(s-h-y)}\right) \\ &\geqslant 2c_8(\delta)\, \overline{\Phi}\left(\frac{a}{\sqrt{(1-\delta)(s-h-y)}}\right) \\ &= 2c_8(\delta)\, \mathbb{P}^{-z-\sqrt{s}}\left(W(s-h-y) > \frac{a}{\sqrt{1-\delta}} - z - \sqrt{s}\right), \end{aligned}$$

the second inequality being a consequence of (2.16). Therefore, for $y \in [0, \delta s]$,

$$\int_0^{s-y} P(s-h-y)\, \mathbb{P}^z(T_{-\sqrt{s}} \in \mathrm{d}h)$$

$$
\begin{aligned}
&\geqslant 2c_8(\delta)\int_0^{s-y}\mathbb{P}^{-z-\sqrt{s}}\left(W(s-h-y)>\frac{a}{\sqrt{1-\delta}}-z-\sqrt{s}\right)\mathbb{P}^z\left(T_{-\sqrt{s}}\in \mathrm{d}h\right)\\
&= 2c_8(\delta)\,\mathbb{P}\left(W(s-y)>\frac{a}{\sqrt{1-\delta}}-z-\sqrt{s}\right)\\
&= 2c_8(\delta)\,\overline{\Phi}\left(\frac{\frac{a}{\sqrt{1-\delta}}-z-\sqrt{s}}{\sqrt{s-y}}\right)\\
&\geqslant 2c_8(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right),
\end{aligned}
$$

(recalling that $z<0$). Plugging this into (2.23), we get

$$
\begin{aligned}
&\mathbb{P}^z\left(a\leqslant W(s)\leqslant a+2\varepsilon\lambda,\ |Y(s)|\leqslant 2\varepsilon\lambda\right)\\
&\geqslant 2c_8(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right)\mathbb{P}^{-\sqrt{s}}\left(T_0\leqslant \delta s,\ |Y(T_0)|\leqslant\sqrt{s}\right)\\
&= c_{11}(\delta)\,\overline{\Phi}\left(\frac{a-z}{(1-\delta)\sqrt{s}}\right),
\end{aligned}
$$

where $c_{11}(\delta):=2c_8(\delta)\,\mathbb{P}^{-1}(T_0\leqslant\delta,\ |Y(T_0)|\leqslant 1)$. This yields (2.21). □

Lemma 2.7 *For $s>0$, $\varepsilon>0$, $\lambda>0$, $(a,z)\in\mathbb{R}^2$ such that $\varepsilon\lambda>2\sqrt{s}$ and $|a|>2\varepsilon\lambda+\sqrt{s}$, we have*

$$
\begin{aligned}
&\mathbb{P}^z(a\leqslant W(s)\leqslant a+2\varepsilon\lambda,\ |Y(s)|\leq 2\varepsilon\lambda)\\
&\geqslant c_{12}(\delta)\,\exp\left(-\frac{a^2}{2(1-\delta)^2 s}\right)\overline{\Phi}\left(\frac{||z|-\sqrt{s}|}{\delta\sqrt{s}}\right),
\end{aligned}
\tag{2.24}
$$

with a constant $c_{12}(\delta)>0$.

Proof: Again, it suffices to treat the case $a>\sqrt{s}$. In this case, we have

$$
\begin{aligned}
&\mathbb{P}^z(a\leqslant W(s)\leqslant a+2\varepsilon\lambda,\ |Y(s)|\leq 2\varepsilon\lambda)\\
&\geqslant \int_0^{\delta s}\mathbb{P}^z(|Y(T_0)|\leq 2\sqrt{s},\ T_0\in\ \mathrm{d}h)\,\mathbb{P}(a\leq W(s-h)\leq a+2\varepsilon\lambda,\ |Y(s-h)|\leq 2\sqrt{s-h}\,)\\
&\geqslant c_{13}(\delta)\,\exp\left(-\frac{a^2}{2(1-\delta)^2 s}\right)\mathbb{P}^z(|Y(T_0)|\leq 2\sqrt{s},\ T_0\leq\delta s),
\end{aligned}
$$

hence (2.24) follows from Lemma 2.3. □

2.3 Increments

Recall the results for the increments of Wiener process (cf. [12]), local time (cf. [5]) and principal value (cf. [6]).

As $T \to \infty$, we have almost surely

$$\sup_{0\leqslant t\leqslant T-a_T}\sup_{0\leqslant s\leqslant a_T}|X(t+s)-X(t)| = \mathcal{O}\left(\sqrt{a_T(\log(T/a_T)+\log\log T)}\right), \tag{2.25}$$

and for fixed T, as $\delta \to 0$ we have almost surely

$$\sup_{0\leqslant t\leqslant T}\sup_{0\leqslant s\leqslant \delta}|X(t+s)-X(t)| = \mathcal{O}(\sqrt{\delta\log(1/\delta)}). \tag{2.26}$$

Here in (2.25) and (2.26) X can be any one of W, L and Y.

3 Proof of Theorem 1.1

The relative compactness in Theorem 1.1 is obvious from Theorems A and B. Our proof will consist of two steps:

(1) With probability one any $(f,g) \notin \mathcal{S}_J^{(2)}$ is not a limit point.

(2) With probability one every $(f,g) \in \mathcal{S}_J^{(2)}$ is a limit point.

Proof of (1): Obviously, if either $f \notin \mathcal{S}$, or $g \notin \mathcal{S}_M$, then (f,g) cannot be a limit point almost surely. From now on we assume that $f \in \mathcal{S}$ and $g \in \mathcal{S}_M$. Now let $x_0 \in (0,1]$ be such that $f(x_0) \neq 0$. Since f is continuous, there exists an interval $(x_1,x_2) \subset [0,1]$ such that $x_0 \in (x_1,x_2]$ and $f(x) \neq 0$ for all $x \in (x_1,x_2)$. If $f(x)$ would be a limit point of $U_t(x)$, then for t big enough, $U_t(x) \neq 0$ for $x \in (x_1,x_2)$, i.e. $W(s) \neq 0$ for $s \in (tx_1,tx_2)$ would hold. Consequently, the local time $L(\cdot)$ would not increase in the interval (tx_1,tx_2), which means that if $g(x_2) > g(x_1)$, then (f,g) cannot be a limit point of (U_t,V_t). So we must have $g'(x_0)=0$, i.e., $f(x)g'(x)=0$ a.e. So we assume from now on that this equation is satisfied. Now we prove that if

$$\int_0^1 \left((f'(x))^2+(g'(x))^2\right)\,\mathrm{d}x > 1, \tag{3.1}$$

then (f,g) cannot be a limit point.

To this end, we need a lemma.

Lemma 3.1 *Let (f,g) be such that $f(x)g'(x)=0$ a.e., and (3.1) holds. Then there exists a partition $x_0=0<x_1<\ldots<x_{k-1}<x_k=1$ of $[0,1]$ such that*

$$\Lambda_k := \sum_{i=1}^{k}\left(\frac{(f_i-f_{i-1})^2}{x_i-x_{i-1}}\mathbb{1}_{\{g_i-g_{i-1}=0\}}+\frac{(g_i-g_{i-1})^2}{x_i-x_{i-1}}\right)>1, \tag{3.2}$$

where $f_i:=f(x_i)$ and $g_i:=g(x_i)$.

Proof: If (3.1) holds, then we can find a partition $x_0=0<x_1<\ldots<x_{j-1}<x_j=1$ of $[0,1]$ such that

$$\sum_{i=1}^{j}\left(\frac{(f_i-f_{i-1})^2}{x_i-x_{i-1}}+\frac{(g_i-g_{i-1})^2}{x_i-x_{i-1}}\right)>1$$

holds. Consider a partition interval (x_{i-1},x_i). We have to consider the following three cases: (i) $f_{i-1}=f_i$, (ii) $f_{i-1}\neq f_i$, and $f_{i-1}f_i\geqslant 0$, (iii) $f_{i-1}\neq f_i$, and $f_{i-1}f_i<0$. In case (i) we can simply write

$$\frac{(f_i-f_{i-1})^2}{x_i-x_{i-1}}=\frac{(f_i-f_{i-1})^2}{x_i-x_{i-1}}\mathbb{1}_{\{g_i-g_{i-1}=0\}}.$$

In case (ii) let $x_i'=\max\{x\leqslant x_i:\ f(x)=f(x_{i-1})\}$ and $x_i''=\min\{x\geqslant x_i':\ f(x)=f(x_i)\}$. (It is possible that $x_i'=x_{i-1}$ or $x_i''=x_i$.) Consider the refinement of the partition by replacing (x_{i-1},x_i) with (x_{i-1},x_i'), (x_i',x_i''), (x_i'',x_i). In the interval (x_i',x_i'') $f(x)$ must strictly be between f_{i-1} and f_i, so $f(x)\neq 0$, hence $g'(x)=0$ for all $x\in(x_i',x_i'')$, thus $g(x_i')=g(x_i'')$. Using the elementary inequality

$$\frac{(a+b)^2}{c+d}\leqslant\frac{a^2}{c}+\frac{b^2}{d},$$

we may write

$$\begin{aligned}
\frac{(f_i-f_{i-1})^2}{x_i-x_{i-1}} &\leqslant \frac{(f(x_i')-f_{i-1})^2}{x_i'-x_{i-1}}+\frac{(f(x_i'')-f(x_i'))^2}{x_i''-x_i'}+\frac{(f_i-f(x_i''))^2}{x_i-x_i''}\\
&= \frac{(f(x_i')-f_{i-1})^2}{x_i'-x_{i-1}}\mathbb{1}_{\{g(x_i')-g_{i-1}=0\}}+\frac{(f(x_i'')-f(x_i'))^2}{x_i''-x_i'}\mathbb{1}_{\{g(x_i'')-g(x_i')=0\}}\\
&+\frac{(f_i-f(x_i''))^2}{x_i-x_i''}\mathbb{1}_{\{g_i-g(x_i'')=0\}}.
\end{aligned}\tag{3.3}$$

In case (iii) let $x_i'=\min\{x\geqslant x_{i-1}:\ f(x)=0\}$ and $x_i''=\max\{x\leqslant x_i:\ f(x)=0\}$. Consider again the refinement of the partition by replacing (x_{i-1},x_i) with (x_{i-1},x_i'), (x_i',x_i''), (x_i'',x_i). In the first and the last of these three intervals $f(x)\neq 0$, hence $g'(x)=0$, thus $g(x_i')=g_{i-1}$ and $g(x_i'')=g_i$. On the other hand, $f(x_i')=f(x_i'')=0$. So we again have (3.3).

This completes the proof of the Lemma. $\square$

Returning to the main course of the proof, choose $\varepsilon > 0$ such that

$$\Lambda_k - 8\varepsilon \sum_{i=1}^{k} \frac{1}{x_i - x_{i-1}} \;>\; 1, \tag{3.4}$$

$$\varepsilon \;<\; \frac{1}{2} \min_{i:\, 1\leqslant i\leqslant k,\, g_i > g_{i-1}} (g_i - g_{i-1}), \tag{3.5}$$

$$\varepsilon \;<\; \frac{1}{2} \min_{i:\, 1\leqslant i\leqslant k,\, f_i \neq f_{i-1}} |f_i - f_{i-1}|. \tag{3.6}$$

Note that $g_i - g_{i-1} - 2\varepsilon > 0$ whenever $g_i - g_{i-1} > 0$, and that $(f_{i-1} - \varepsilon, f_{i-1} + \varepsilon)$ and $(f_i - \varepsilon, f_i + \varepsilon)$ are disjoint intervals as long as $f_i \neq f_{i-1}$. We may assume that $|f_i| \leqslant 1$, $g_i \leqslant 1$, $i = 1, \dots, k$, since otherwise (f, g) cannot be a limit point according to the usual law of the iterated logarithm.

Consider the event

$$A_t := \bigcap_{i=1}^{k} \{ f_i - \varepsilon \leqslant U_t(x_i) \leqslant f_i + \varepsilon,\; g_i - \varepsilon \leqslant V_t(x_i) \leqslant g_i + \varepsilon \}.$$

If we write, for $1 \leqslant i \leqslant k$,

$$\begin{aligned}
s_i &:= x_i t, \qquad (s_0 := 0), \\
a_i &:= (f_i - \varepsilon)(2t \log\log t)^{1/2}, \quad b_i := (f_i + \varepsilon)(2t \log\log t)^{1/2}, \\
\alpha_i &:= (g_i - g_{i-1} - 2\varepsilon)^+ (2t \log\log t)^{1/2}, \quad \beta_i := (g_i - g_{i-1} + 2\varepsilon)(2t \log\log t)^{1/2},
\end{aligned}$$

then by the conditional independence of $\{L(s_i) - L(s_{i-1})\}_{1\leqslant i\leqslant k}$ given $\{W(s_1), \cdots, W(s_k)\}$, we obtain:

$$\begin{aligned}
\mathbb{P}(A_t) &= \mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant L(s_i) - L(s_{i-1}) \leqslant \beta_i,\, i = 1, \dots, k) \\
&= \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} \prod_{i=1}^{k} \mathbb{P}(\alpha_i \leqslant L(s_i) - L(s_{i-1}) \leqslant \beta_i \,|W(s_1) = z_1, \dots, W(s_k) = z_k) \times \\
&\qquad \times \mathbb{P}(W(s_1) \in \mathrm{d}z_1, \dots, W(s_k) \in \mathrm{d}z_k).
\end{aligned} \tag{3.7}$$

According to (2.1) and (2.2) (with the same notation there), the integrand above is equal to

$$\prod_{i=1}^{k} \Bigg[\left(\int_{\alpha_i}^{\beta_i} p_{s_i - s_{i-1}}(y_i; z_{i-1}, z_i)\, \mathrm{d}y_i \right) \mathbb{1}_{\{\alpha_i > 0\}} + \left(\int_{0}^{\beta_i} p_{s_i - s_{i-1}}(y_i; z_{i-1}, z_i)\, \mathrm{d}y_i + q_{s_i - s_{i-1}}(z_{i-1}, z_i) \right) \mathbb{1}_{\{\alpha_i = 0\}} \Bigg].$$

We now estimate $p_t(y;z_1,z_2)$ and $q_t(z_1,z_2)$. If $0<\alpha\leqslant y\leqslant\beta\leqslant\beta^*$, $|z_1|\leqslant\beta^*$ and $|z_2|\leqslant\beta^*$ (for some β^*), then

$$\begin{aligned}
p_t(y;z_1,z_2) &\leqslant \frac{\beta+2\beta^*}{t}\varphi_t(\alpha),\\
\int_\alpha^\beta p_t(y;z_1,z_2)\,\mathrm{d}y &\leqslant \frac{3(\beta^*)^2}{t}\varphi_t(\alpha),\\
\int_0^\beta p_t(y;z_1,z_2)\,\mathrm{d}y &\leqslant \frac{3(\beta^*)^2}{t}\varphi_t(|z_1|+|z_2|),
\end{aligned}$$

where $\varphi_t(\cdot)$ is defined in (2.5). On the other hand, if $3(\beta^*)^2\geqslant t$, then

$$q_t(z_1,z_2)\leqslant\frac{3(\beta^*)^2}{t}\left(\varphi_t(z_1-z_2)-\varphi_t(|z_1|+|z_2|)\right).$$

Consequently,

$$\begin{aligned}
&\left(\int_\alpha^\beta p_t(y;z_1,z_2)\,\mathrm{d}y\right)\mathbb{1}_{\{\alpha>0\}}+\left(\int_0^\beta p_t(y;z_1,z_2)\,\mathrm{d}y+q_t(z_1,z_2)\right)\mathbb{1}_{\{\alpha=0\}}\\
\leqslant\;& \frac{3(\beta^*)^2}{t}\left(\varphi_t(\alpha)\mathbb{1}_{\{\alpha>0\}}+\varphi_t(z_2-z_1)\mathbb{1}_{\{\alpha=0\}}\right)\\
=\;& \frac{3(\beta^*)^2}{t^{3/2}\sqrt{2\pi}}\exp\left(-\frac{\alpha^2\mathbb{1}_{\{\alpha>0\}}+(z_2-z_1)^2\mathbb{1}_{\{\alpha=0\}}}{2t}\right)\\
=\;& \frac{3(\beta^*)^2}{t^{3/2}\sqrt{2\pi}}\exp\left(-\frac{\alpha^2+(z_2-z_1)^2\mathbb{1}_{\{\alpha=0\}}}{2t}\right).
\end{aligned}$$

Putting this estimate into (3.7) and using that $\alpha_i\leqslant 2(t\log\log t)^{1/2}$, $\beta_i\leqslant 2(t\log\log t)^{1/2}$, we obtain

$$\begin{aligned}
\mathbb{P}(A_t) \leqslant\;& \frac{c_{14}\,(\log\log t)^k}{t^{k/2}}\exp\left(-\frac{1}{2t}\sum_{i=1}^k\frac{\alpha_i^2}{(x_i-x_{i-1})}\right)\\
&\times\int_{a_1}^{b_1}\cdots\int_{a_k}^{b_k}\exp\left(-\frac{1}{2t}\sum_{i=1}^k\frac{(z_i-z_{i-1})^2}{(x_i-x_{i-1})}\mathbb{1}_{\{g_i=g_{i-1}\}}\right)\mathrm{d}z_1\ldots\mathrm{d}z_k,
\end{aligned}$$

with some constant c_{14} depending on ε, and $x_1,\ldots,x_k$ but not on t. Recall Λ_k from (3.2). Using the estimates $\alpha_i^2\geqslant(2t\log\log t)((g_i-g_{i-1})^2-4\varepsilon)$ and $(z_i-z_{i-1})^2\geqslant(2t\log\log t)((f_i-f_{i-1})^2-4\varepsilon)$, we obtain by means of (3.4) that

$$\begin{aligned}
\mathbb{P}(A_t) &\leqslant c_{15}\,(\log\log t)^{3k/2}\exp\left(-(\log\log t)\left(\Lambda_k-8\varepsilon\sum_{i=1}^k\frac{1}{x_i-x_{i-1}}\right)\right)\\
&= c_{15}\,(\log\log t)^{3k/2}\exp\left(-(1+\delta)\log\log t\right),
\end{aligned}$$

for some $\delta > 0$.

Let $t = t_n = \exp(n/(\log n))$. Then $\sum_n \mathbb{P}(A_{t_n}) < \infty$. By the Borel–Cantelli lemma,

$$\liminf_{n\to\infty} d\left((U_{t_n}, V_{t_n}),\, (f,g)\right) \geqslant \varepsilon \qquad \text{a.s.} \tag{3.8}$$

On the other hand, we infer from increment results in Section 2.3 that

$$\lim_{n\to\infty} \sup_{t\in[t_n,t_{n+1}]} \sup_{x\in[0,1]} |U_t(x) - U_{t_n}(x)| = 0 \qquad \text{a.s.,} \tag{3.9}$$

$$\lim_{n\to\infty} \sup_{t\in[t_n,t_{n+1}]} \sup_{x\in[0,1]} |V_t(x) - V_{t_n}(x)| = 0 \qquad \text{a.s.,} \tag{3.10}$$

Combining (3.9)–(3.10) with (3.8) gives that

$$\liminf_{t\to\infty} d\left((U_t, V_t),\, (f,g)\right) \geqslant \varepsilon \qquad \text{a.s.}$$

for some $\varepsilon > 0$.

Thus we proved that if $(f,g) \notin \mathcal{S}_J^{(2)}$, then it is not a limit point with probability one, i.e. (f,g) has an open ball of radius ε not containing (U_t, V_t) for large enough t. However the exceptional ω-set of probability zero may depend on (f,g). Now we prove that the totality of these exceptional ω-sets is still of probability zero. Denote the complement of $\mathcal{S}_J^{(2)}$ by $\mathcal{D}$ and for each $(f,g) \in \mathcal{D}$ consider the open balls defined above. Their union covers $\mathcal{D}$ and being $\mathcal{C}^{(2)}$ separable, we can select a countable subcover (cf. e.g. [2], p. 217). The union of exceptional ω-sets belonging to this countable subcover is still of probability zero. We call the complement of this last set of probability zero as our universal ω-set. Each $(f,g) \in \mathcal{D}$ has a neighborhood which is completely contained in one of the elements of the countable subcover, hence on the universal set this neighborhood for large enough t does not contain (U_t, V_t), i.e. (f,g) is not a limit point. This completes the proof of (1).

Proof of (2): Assume that $(f,g) \in \mathcal{S}_J^{(2)}$ with strict inequality in the integral criterion, i.e.

$$\int_0^1 \left((f'(x))^2 + (g'(x))^2\right) \mathrm{d}x < 1.$$

For given $\varepsilon_1 > 0$, choose a partition $x_0 = 0 < x_1 \ldots < x_k = 1$ of the interval $[0,1]$ such that

$$\begin{aligned}
\sup_{1\leqslant i\leqslant k} (x_i - x_{i-1}) &\leqslant \varepsilon_1^2, \\
\sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i]} |f(x) - f_i| &\leqslant \varepsilon_1, \\
\sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i]} |g(x) - g_i| &\leqslant \varepsilon_1,
\end{aligned}$$

where $f_i = f(x_i)$, $g_i = g(x_i)$. We may assume that if $g_{i-1} \neq g_i$, then $f_{i-1} = f_i = 0$. Otherwise if it happens that $g_{i-1} \neq g_i$ but either $f_{i-1} \neq 0$ or $f_i \neq 0$ (or both), then we can choose $x' = \min\{x : x > x_{i-1}, f(x) = 0\}$, $x'' = \max\{x : x < x_i, f(x) = 0\}$. We must have $g(x') = g_{i-1}$ and $g(x'') = g_i$ so by refining the original partition by inserting new points x', x'', the new partition satisfies the above assumption. Since

$$\frac{(f(x_i) - f(x_{i-1})^2}{x_{i-1} - x_i} \leqslant \int_{x_{i-1}}^{x_i} (f'(x))^2 \, \mathrm{d}x, \qquad \frac{(g(x_i) - g(x_{i-1}))^2}{x_{i-1} - x_i} \leqslant \int_{x_{i-1}}^{x_i} (g'(x))^2 \, \mathrm{d}x,$$

(cf. for example [13], p. 52), we have $\Lambda_k < 1$, where Λ_k is given by (3.2). Then choose $\varepsilon > 0$ such that

$$\Lambda_k + 21\varepsilon \sum_{i=1}^{k} \frac{1}{x_i - x_{i-1}} < 1,$$

and that

$$(f_i - \varepsilon)(f_i + \varepsilon) > 0 \quad \text{if} \quad f_i \neq 0.$$

Now define the events $E_t = \cap_{i=1}^k E_t^{(i)}$ with

$$E_t^{(i)} = \{a_i \leqslant W(s_i) \leqslant b_i, \, \alpha_i \leqslant L(s_i) - L(s_{i-1}) \leqslant \beta_i \},$$

$s_i = tx_i$,

$$a_i = (f_i - \varepsilon)(2t \log\log t)^{1/2}, \quad b_i = (f_i + \varepsilon)(2t \log\log t)^{1/2},$$

$$\alpha_i = (g_i - g_{i-1} - \varepsilon)^+ (2t \log\log t)^{1/2}, \quad \beta_i = (g_i - g_{i-1} + \varepsilon)(2t \log\log t)^{1/2}.$$

This time we bound $\mathbb{P}(E_t)$ from below.

We get

$$\int_\alpha^\beta p_s(y; z_1, z_2) \, \mathrm{d}y \geqslant \frac{(\beta - \alpha)^2}{2s} \varphi_s(\beta + |z_1| + |z_2|) \geqslant \varphi_s(\beta + |z_1| + |z_2|),$$

provided that $(\beta - \alpha)^2 \geqslant 2s$. But $(\beta_i - \alpha_i)^2 \geqslant 2\varepsilon^2 t \log\log t \geqslant 2t(x_i - x_{i-1}) = 2(s_i - s_{i-1})$ for all $i = 1, \ldots, k$ if t is large enough. Therefore we have (cf. 3.7)

$$\mathbb{P}(E_t) \geqslant \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} \prod_{i=1}^{k} \Big(\varphi_{s_i - s_{i-1}}(\beta_i + |z_{i-1}| + |z_i|)$$
$$+ \big(\varphi_{s_i - s_{i-1}}(z_i - z_{i-1}) - \varphi_{s_i - s_{i-1}}(|z_i| + |z_{i-1}|)\big) \, \mathbb{1}_{\{g_i = g_{i-1}\}} \Big) \, \mathrm{d}z_1 \ldots \mathrm{d}z_k.$$

for all t large enough.

Now we consider 3 cases.

Case 1: $g_i > g_{i-1}$, $f_{i-1} = f_i = 0$. We have

$$\begin{aligned}\varphi_{t(x_i-x_{i-1})}(\beta_i + |z_{i-1}| + |z_i|) &\geqslant \frac{1}{\sqrt{2\pi t(x_i - x_{i-1})}} \exp\left(-\frac{(g_i - g_{i-1} + 3\varepsilon)^2}{x_i - x_{i-1}} \log\log t\right) \\ &\geqslant \frac{1}{2\sqrt{2\pi t(x_i - x_{i-1})}} \exp\left(-\frac{(g_i - g_{i-1})^2 + 21\varepsilon}{x_i - x_{i-1}} \log\log t\right).\end{aligned}$$

Case 2: $g_i = g_{i-1}$, $f_i f_{i-1} \leqslant 0$. In this case $\beta_i = \varepsilon(2t\log\log t)^{1/2}$, $|z_i| \leqslant (|f_i|+\varepsilon)(2t\log\log t)^{1/2}$, $|z_{i-1}| \leqslant (|f_{i-1}| + \varepsilon)(2t\log\log t)^{1/2}$, hence we have

$$\begin{aligned}\varphi_{t(x_i-x_{i-1})}(\beta_i + |z_{i-1}| + |z_i|) &\geqslant \frac{1}{\sqrt{2\pi t(x_i - x_{i-1})}} \exp\left(-\frac{(|f_{i-1}| + |f_i| + 3\varepsilon)^2}{x_i - x_{i-1}} \log\log t\right) \\ &\geqslant \frac{1}{2\sqrt{2\pi t(x_i - x_{i-1})}} \exp\left(-\frac{(f_i - f_{i-1})^2 + 21\varepsilon}{x_i - x_{i-1}} \log\log t\right).\end{aligned}$$

Case 3: $g_i = g_{i-1}$, $f_i f_{i-1} > 0$. Assume $f_i > 0$, $f_{i-1} > 0$, the other case being similar. In this case we have also $f_i - \varepsilon > 0$, $f_{i-1} - \varepsilon > 0$, so that $z_i > 0$, $z_{i-1} > 0$. Hence

$$\begin{aligned}&\varphi_{t(x_i-x_{i-1})}(z_i - z_{i-1}) - \varphi_{t(x_i-x_{i-1})}(z_i + z_{i-1}) \\ =\ & \frac{1}{\sqrt{2\pi t(x_i - x_{i-1})}} \exp\left(-\frac{(z_i - z_{i-1})^2}{2t(x_i - x_{i-1})}\right)\left(1 - \exp\left(-\frac{2z_i z_{i-1}}{t(x_i - x_{i-1})}\right)\right) \\ \geqslant\ & \frac{1}{2\sqrt{2\pi t(x_i - x_{i-1})}} \exp\left(-\frac{(f_i - f_{i-1})^2 + 21\varepsilon}{x_i - x_{i-1}} \log\log t\right),\end{aligned}$$

if t is large enough.

Assembling these 3 cases we get

$$\mathbb{P}(E_t) \geqslant c_{16} (\log\log t)^{k/2} \exp\left(-\left(\Lambda_k + 35\varepsilon \sum_{i=1}^{k} \frac{1}{x_i - x_{i-1}}\right) \log\log t\right),$$

where Λ_k is given by (3.2).

Hence we obtain that

$$\mathbb{P}(E_t) \geqslant \exp(-(1-\delta)\log\log t) \tag{3.11}$$

for some $0 < \delta < 1$.

Now let $t_i = \exp(7i \log i)$, $i = 1, 2, \ldots$ and define

$$\eta_0 = 0, \quad T_i = \eta_{i-1} + t_i, \quad \eta_i = \inf\{t :\ t > T_i,\ W(t) = 0\},\ i = 1, 2, \ldots$$

It was shown in [8] that for large n we have almost surely

$$t_n \leqslant T_n \leqslant t_n\left(1 + \frac{1}{n}\right).$$

Define

$$
\begin{aligned}
\widehat{W}^{(n)}(t) &= W(t+\eta_{n-1}), \quad t \geqslant 0, \\
\widehat{L}^{(n)}(t) &= L(t+\eta_{n-1}) - L(\eta_{n-1}), \quad t \geqslant 0, \\
\widehat{U}^{(n)}(x) &= \frac{\widehat{W}^{(n)}(xt_n)}{\sqrt{2t_n \log\log t_n}}, \quad x \in [0,1], \\
\widehat{V}^{(n)}(x) &= \frac{\widehat{L}^{(n)}(xt_n)}{\sqrt{2t_n \log\log t_n}}, \quad x \in [0,1].
\end{aligned}
$$

Now let $x_0 = 0 < x_1 < \ldots < x_k$ be a partition as before and consider the events $\widehat{E}_n = \cap_{i=1}^k \widehat{E}_n^{(i)}$ with

$$
\widehat{E}_n^{(i)} = \{\widehat{a}_i \leqslant \widehat{W}^{(n)}(\widehat{s}_i) \leqslant \widehat{b}_i,\ \widehat{\alpha}_i \leqslant \widehat{L}^{(n)}(\widehat{s}_i) - \widehat{L}^{(n)}(\widehat{s}_{i-1}) \leqslant \widehat{\beta}_i\},
$$

$\widehat{s}_i = x_i t_n$,

$$
\widehat{a}_i = (f_i - \varepsilon)(2t_n \log\log t_n)^{1/2}, \quad \widehat{b}_i = (f_i + \varepsilon)(2t_n \log\log t_n)^{1/2},
$$

$$
\widehat{\alpha}_i = (g_i - g_{i-1} - \varepsilon)^+(2t_n \log\log t_n)^{1/2}, \quad \widehat{\beta}_i = (g_i - g_{i-1} + \varepsilon)(2t_n \log\log t_n)^{1/2}.
$$

Since $\mathbb{P}(\widehat{E}_n) = \mathbb{P}(E_{t_n})$, it follows from (3.11) that $\sum_n \mathbb{P}(\widehat{E}_n) = \infty$ and since $\widehat{E}_n$ are independent, we have by the Borel–Cantelli lemma $\mathbb{P}(\widehat{E}_n \text{ i.o.}) = 1$. Since $\varepsilon > 0$ is arbitrary, this implies

$$
\begin{aligned}
\liminf_{n\to\infty} \sup_{1\leqslant i\leqslant k} |\widehat{U}^{(n)}(x_i) - f(x_i)| &= 0 \qquad \text{a.s.} \\
\liminf_{n\to\infty} \sup_{1\leqslant i\leqslant k} |\widehat{V}^{(n)}(x_i) - g(x_i)| &= 0 \qquad \text{a.s.}
\end{aligned}
$$

Again, from the increment results in Subsection 2.3 it follows that

$$
\begin{aligned}
\limsup_{n\to\infty} \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i)} |\widehat{U}^{(n)}(x_{i-1}) - \widehat{U}^{(n)}(x)| &\leqslant \varepsilon_1 \qquad \text{a.s.} \\
\limsup_{n\to\infty} \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i)} |\widehat{V}^{(n)}(x_{i-1}) - \widehat{V}^{(n)}(x)| &\leqslant \varepsilon_1 \qquad \text{a.s.}
\end{aligned}
$$

Since $\varepsilon_1 > 0$ is arbitrary, these yield

$$
\liminf_{n\to\infty} d\left((\widehat{U}^{(n)}, \widehat{V}^{(n)}),\ (f,g)\right) = 0 \qquad \text{a.s.}
$$

On the other hand, the increment results in Subsection 2.3 once again yields that, as $n \to \infty$, $d((\widehat{U}^{(n)}, \widehat{V}^{(n)}),\ (U_{T_n}, V_{T_n}))$ converges to 0 almost surely. Therefore,

$$
\liminf_{n\to\infty} d\left((U_{T_n}, V_{T_n}),\ (f,g)\right) = 0 \qquad \text{a.s.,}
$$

Hence, (f,g) is a limit point of (U_t, V_t) with probability 1.

To complete the proof of Theorem 1.1, we have to show that there exists an ω-set of probability one for which every $(f,g) \in S_J^{(2)}$ is a limit point.

First we show that there exists a countable dense subset $K \subset S_J^{(2)}$. For any $(f,g) \in S_J^{(2)}$ and $\varepsilon > 0$, as before, choose a partition $x_0 = 0 < x_1 < \ldots < x_{k-1} < x_k = 1$ such that

$$\sup_{x_{i-1} \leqslant x \leqslant x_i} |f(x) - f(x_i)| \leqslant \varepsilon, \qquad \sup_{x_{i-1} \leqslant x \leqslant x_i} |g(x) - g(x_i)| \leqslant \varepsilon$$

and $g(x_{i-1}) \neq g(x_i)$ implies $f(x_{i-1}) = f(x_i) = 0$. Define $(\widetilde{f}, \widetilde{g}) \in S_J^{(2)}$ such that $\widetilde{f}(x_i) = f(x_i)$, $\widetilde{g}(x_i) = g(x_i)$, $i = 1, 2, \ldots, k$ and let $\widetilde{f}$ and $\widetilde{g}$ be linear in between. Then

$$d((f,g),\, (\widetilde{f}, \widetilde{g})) < 2\sqrt{2}\varepsilon,$$

meaning that the set of pairs (f,g), where both f and g are piecewise linear (with the same cut-off points), is dense. It can be seen that one can choose a countable dense subset $K = \{(f_n, g_n)\}_{n=1}^{\infty}$ (for example by taking all x_i, $f_n(x_i)$, $g_n(x_i)$ rational) such that

$$\int_0^1 (f_n'(x))^2 + (g_n'(x))^2 \, \mathrm{d}x < 1.$$

It follows that there exists an ω-set of probability one such that all $(f_n, g_n) \in K$ are limit points. Next we show that for this ω-set every $(f,g) \in S_J^{(2)}$ is a limit point. Since K is dense, for each n we find $(f_n, g_n) \in K$ such that

$$d((f,g),\, (f_n, g_n)) < \frac{1}{n}$$

and since (f_n, g_n) is a limit point, we can find t_n such that $d((f_n, g_n),\, (U_{t_n}, V_{t_n})) < \frac{1}{n}$. Hence $d((f,g),\, (U_{t_n}, V_{t_n})) < \frac{2}{n}$. Consequently,

$$\lim_{n \to \infty} (U_{t_n}, V_{t_n}) = (f,g),$$

i.e., (f,g) is a limit point.

This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.3

According to (2.25) for Y,

$$\lim_{\delta\to 0}\sup_{t\geqslant 1}\sup_{0\leqslant x,x'\leqslant 1,\,|x-x'|\leqslant\delta}|Z_t(x)-Z_t(x')|\to 0,\qquad \text{a.s.}$$

Now the relative compactness of $\{Z_t\}_{t\geqslant 1}$ in $\mathcal{C}$ follows from the Arzelà–Ascoli theorem. Hence $\{(U_t, Z_t)\}$ is relatively compact in $\mathcal{C}^{(2)}$.

As in the case of Theorem 1.1, our proof will consist of two steps:

(1) With probability one any $(f,g)\notin\widetilde{\mathcal{S}}_J^{(2)}$ is not a limit point.

(2) With probability one every $(f,g)\in\widetilde{\mathcal{S}}_J^{(2)}$ is a limit point.

Proof of (1): Let $x_0\in(0,1]$ be a point, where $f(x_0)\neq 0$. Since f is continuous, there exists an interval $(x_1,x_2)\subset[0,1]$ such that $x_0\in(x_1,x_2]$ and $f(x)\neq 0$ for all $x\in(x_1,x_2)$. We show that if (f,g) is a limit point, then g is constant in (x_1,x_2). Since (f,g) is a limit point, there exists a sequence $\{t_n\}_{n\geqslant 1}$ such that

$$|W(xt_n)|\geqslant c_{17}\sqrt{2t_n\log\log t_n},\quad x\in(x_1,x_2)$$

for some $c_{17}>0$ and for every $x\in(x_1,x_2)$

$$\begin{aligned}\left|\frac{Y(xt_n)-Y(x_0t_n)}{\sqrt{8t_n\log\log t_n}}\right| &= \frac{1}{\sqrt{8t_n\log\log t_n}}\left|\int_{x_0t_n}^{xt_n}\frac{\mathrm{d}s}{W(s)}\right|\\ &\leqslant \frac{|xt_n-x_0t_n|}{4c_{17}\,t_n\log\log t_n}\to 0=g(x)-g(x_0),\end{aligned}$$

as $n\to\infty$. So $g(x)=g(x_0)$ for every $x\in(x_1,x_2)$. So if (f,g) is a limit point and g is absolutely continuous (which is not guaranteed so far), then we must have $f(x)g'(x)=0$ a.e.

Now if either g is not absolutely continuous, or g is absolutely continuous but $\int_0^1((f'(x))^2+(g'(x))^2)\,\mathrm{d}x>1$, by Lemma 3.1 we can find a partition $x_0=0<x_1\ldots<x_{k-1}<x_k=1$ and $\delta>0$ such that

$$\widetilde{\Lambda}_k:=\sum_{i=1}^{k}\left(\frac{(f_i-f_{i-1})^2}{x_i-x_{i-1}}\mathbb{1}_{\{g_i-g_{i-1}=0\}}+\frac{8}{8+\delta}\frac{(g_i-g_{i-1})^2}{x_i-x_{i-1}}\right)>1.$$

Then choose $\varepsilon>0$ such that

$$\widetilde{\Lambda}_k-20\varepsilon\sum_{i=1}^{k}\frac{1}{x_i-x_{i-1}}>1,$$

$$(f_{i-1}-\varepsilon, f_{i-1}+\varepsilon) \quad \text{and} \quad (f_i-\varepsilon, f_i+\varepsilon) \quad \text{are disjoint if } f_i \neq f_{i-1},$$
$$|g_i - g_{i-1}| > 6\varepsilon \text{ if } g_i \neq g_{i-1}.$$

Here $f_i = f(x_i)$ and $g_i = g(x_i)$, $i = 1, \ldots, k$. We may also assume that $|f_i - f_{i-1}| \leqslant 1$ and $|g_i - g_{i-1}| \leqslant 1$, $i = 1, \ldots, k$, otherwise (f, g) cannot be a limit point by the usual law of the iterated logarithm.

Define the events

$$\begin{aligned} \widetilde{A}_t^{(i)} &= \{f_i - \varepsilon \leqslant U_t(x_i) \leqslant f_i + \varepsilon,\ g_i - g_{i-1} - 2\varepsilon \leqslant Z_t(x_i) - Z_t(x_{i-1}) \leqslant g_i - g_{i-1} + 2\varepsilon\} \\ &= \{a_i \leqslant W(s_i) \leqslant b_i,\ \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i\} \end{aligned}$$

with $s_i = x_i t$ and

$$a_i = (f_i - \varepsilon)(2t \log\log t)^{1/2}, \quad b_i = (f_i + \varepsilon)(2t \log\log t)^{1/2},$$

$$\alpha_i = (g_i - g_{i-1} - 2\varepsilon)2(2t \log\log t)^{1/2}, \quad \beta_i = (g_i - g_{i-1} + 2\varepsilon)2(2t \log\log t)^{1/2}.$$

It follows from Lemma 2.1 putting $\lambda = (2t \log\log t)^{1/2}$ there

$$\mathbb{P}(\widetilde{A}_t^{(i)} \mid W(s_{i-1}) = z_{i-1}) \leqslant \sqrt{\frac{2 \log\log t}{x_i - x_{i-1}}} \exp\left(-\frac{(f_i - f_{i-1})^2 - 8\varepsilon}{x_i - x_{i-1}} \log\log t\right)$$

and if $g_i \neq g_{i-1}$, then

$$\mathbb{P}(\widetilde{A}_t^{(i)} \mid W(s_{i-1}) = z_{i-1}) \leqslant c_{18} \exp\left(-\frac{(g_i - g_{i-1})^2 - 20\varepsilon}{(8+\delta)(x_i - x_{i-1})} 8 \log\log t\right)$$

with some $c_{18} > 0$. So for large enough t we have

$$\begin{aligned} \mathbb{P}(\widetilde{A}_t^{(i)} \mid W(s_{i-1}) = z_{i-1}) \ &\leqslant\ c_{19} \sqrt{\frac{\log\log t}{x_i - x_{i-1}}} \left[\exp\left(-\frac{(f_i - f_{i-1})^2 - 8\varepsilon}{x_i - x_{i-1}} \log\log t\right) \mathbb{1}_{\{g_i = g_{i-1}\}}\right. \\ &\quad + \left.\exp\left(-\frac{(g_i - g_{i-1})^2 - 20\varepsilon}{(8+\delta)(x_i - x_{i-1})} 8 \log\log t\right) \mathbb{1}_{\{g_i \neq g_{i-1}\}}\right]. \end{aligned}$$

It follows that for all large t and some constants $c_{20} > 0$ and $\widetilde{\delta} > 0$,

$$\begin{aligned} \mathbb{P}(\cap_{i=1}^k \widetilde{A}_t^{(i)}) \ &\leqslant\ c_{20} (\log\log t)^{3k/2} \exp\left(-\left(\widetilde{\Lambda}_k - 20\varepsilon \sum_{i=1}^k \frac{1}{x_i - x_{i-1}}\right) \log\log t\right) \\ &\leqslant\ \exp(-(1 + \widetilde{\delta}) \log\log t). \end{aligned}$$

From here, we can complete the proof of part (i) just as in part (i) of the proof of Theorem 1.1.

Proof of (2): Assume that $(f,g) \in \widetilde{\mathcal{S}}_J^{(2)}$ with strict inequality in the integral criterion, i.e.,

$$\int_0^1 ((f'(x))^2 + (g'(x))^2)\,\mathrm{d}x < 1.$$

For given $\varepsilon_1 > 0$, choose a partition $x_0 = 0 < x_1 \ldots < x_k = 1$ of the interval $[0,1]$ such that

$$\sup_{1\leqslant i\leqslant k} (x_i - x_{i-1}) \leqslant \varepsilon_1^2,$$

$$\sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i]} |f(x) - f_i| \leqslant \varepsilon_1, \qquad \sup_{1\leqslant i\leqslant k} \sup_{x\in[x_{i-1},x_i]} |g(x) - g_i| \leqslant \varepsilon_1,$$

where $f_i = f(x_i)$, $g_i = g(x_i)$. Exactly as in Theorem 1.1, we may assume that $f_{i-1} = f_i = 0$ whenever $g_{i-1} \neq g_i$. Also, as in the proof of Theorem 1.1, since

$$\frac{(f(x_i) - f(x_{i-1})^2}{x_{i-1} - x_i} \leqslant \int_{x_{i-1}}^{x_i} (f'(x))^2\,\mathrm{d}x, \qquad \frac{(g(x_i) - g(x_{i-1}))^2}{x_{i-1} - x_i} \leqslant \int_{x_{i-1}}^{x_i} (g'(x))^2\,\mathrm{d}x,$$

we have

$$\bar{\Lambda}_k := \sum_{i=1}^k \frac{(f_i - f_{i-1})^2 + (g_i - g_{i-1})^2}{x_i - x_{i-1}} < 1. \tag{4.1}$$

Now choose $0 < \delta < 1$ such that $\bar{\Lambda}_k < (1-\delta)^2$ and then choose $\varepsilon > 0$ such that

$$\Gamma := \frac{\bar{\Lambda}_k}{(1-\delta)^2} + \left(\frac{20\varepsilon}{(1-\delta)^2} + \frac{2\varepsilon}{\delta^2}\right) \sum_{i=1}^k \frac{1}{x_i - x_{i-1}} < 1 \tag{4.2}$$

and $5\varepsilon < |f_i|$, $i = 1, 2, \ldots, k$.

Introduce the notations $\lambda = (2t \log\log t)^{1/2}$, $s_i = tx_i$,

$$a_i = (f_i - \varepsilon)\lambda, \qquad b_i = (f_i + \varepsilon)\lambda,$$

$$\alpha_i = 2(g_i - g_{i-1} - \varepsilon)\lambda, \qquad \beta_i = 2(g_i - g_{i-1} + \varepsilon)\lambda.$$

By using the strong Markov property of the Wiener process, it is readily seen that

$$\begin{aligned} &\mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\, i = 1, 2, \ldots, k) \\ \geqslant\ &\mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\, i = 1, 2, \ldots, k-1) \times \\ &\times \inf_{a_{k-1}\leqslant z_{k-1}\leqslant b_{k-1}} \mathbb{P}(a_k \leqslant W(s_k) \leqslant b_k,\, \alpha_k \leqslant Y(s_k) - Y(s_{k-1}) \leqslant \beta_k \,|\, W(s_{k-1}) = z_{k-1}). \end{aligned}$$

Iterating this argument we can see that

$$\begin{aligned} &\mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\, i = 1, 2, \ldots, k) \\ \geqslant\ &\prod_{i=1}^k \inf_{a_{i-1}\leqslant z_{i-1}\leqslant b_{i-1}} \mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\, \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i \,|\, W(s_{i-1}) = z_{i-1}). \end{aligned} \tag{4.3}$$

Next we show that for $i = 1, 2, \ldots, k$ we have

$$\begin{aligned} &\mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\ \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i \mid W(s_{i-1}) = z_{i-1}) \\ \geqslant\ & \frac{c_{21}(\delta)}{(\log\log t)^{1/2}} \exp\left(-\left(\frac{(f_i - f_{i-1})^2 + (g_i - g_{i-1})^2 + 20\varepsilon}{(1-\delta)^2(x_i - x_{i-1})} + \frac{2\varepsilon}{\delta^2(x_i - x_{i-1})}\right)\log\log t\right) \end{aligned} \tag{4.4}$$

with some $c_{21}(\delta) > 0$.

To see (4.4) we apply Lemmas 2.4–2.7 with $s = s_i - s_{i-1} = t(x_i - x_{i-1})$, $\lambda = (2t\log\log t)^{1/2}$ and t large enough and use the inequalities $|f_i - f_{i-1}| \leqslant 1$, $|g_i - g_{i-1}| \leqslant 1$, $\varepsilon < 1$.

(1) In case $f_i = f_{i-1} = 0$, apply Lemma 2.4 with $\alpha = (g_i - g_{i-1} - \varepsilon)\lambda$, $|z| \leqslant \varepsilon\lambda$ and observe that by (2.15), $\overline{\Phi}$ gives a constant$\times(\log\log t)^{-1/2}$ factor in front of the exponent.

(2) In case $g_i = g_{i-1}$, $f_i f_{i-1} > 0$, apply Lemma 2.5 with $a = (f_i - \varepsilon)\lambda$ and use $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$.

(3) In case $g_i = g_{i-1}$, $f_i f_{i-1} < 0$, apply Lemma 2.6 with $a = (f_i - \varepsilon)\lambda$ and use $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$.

(4) In case $g_i = g_{i-1}$, $f_i = 0$, $f_{i-1} \neq 0$, apply Lemma 2.4 with $\alpha = -2\varepsilon\lambda$, use that $|z - f_{i-1}\lambda| \leqslant \varepsilon\lambda$ and replace δ by $1 - \delta$.

(5) In case $g_i = g_{i-1}$, $f_i = 0$, $f_{i-1} \neq 0$, apply Lemma 2.7 with $a = (f_i - \varepsilon)\lambda$, $|z| \leqslant \varepsilon\lambda$.

Assembling all these estimations, (4.4) follows. This combined with (4.3) gives

$$\begin{aligned} &\mathbb{P}(a_i \leqslant W(s_i) \leqslant b_i,\ \alpha_i \leqslant Y(s_i) - Y(s_{i-1}) \leqslant \beta_i,\ i = 1, 2, \ldots, k) \\ \geqslant\ & \frac{(c_{21}(\delta))^k}{(\log\log t)^{k/2}} \exp(-\Gamma \log\log t), \end{aligned}$$

where $\Gamma < 1$ is given by (4.2). Now the proof can be completed exactly the same way as that of Theorem 1.1. □

5 Proof of Corollaries

The proof of Corollary 1.4 is obvious. To show Corollaries 1.5 and 1.6 we need the following lemma.

Lemma 5.1 *If f and g are absolutely continuous functions and $f(x)g'(x) = 0$ a.e., then*

$$\int_0^1 (f'(x))^2\, \mathbb{1}_{\{g'(x)\neq 0\}}\, \mathrm{d}x = 0. \tag{5.1}$$

Proof: Let

$$\mathcal{A} = \{x \in [0,1] : \ f(x) = 0, \ f'(x) \neq 0\}.$$

For each $x \in \mathcal{A}$, there exists $\delta_x > 0$ such that $f(y) \neq 0$ for all $y \in (x - \delta_x, \ x + \delta_x) \backslash \{x\}$. The intervals $\{(x - \delta_x, \ x + \delta_x)\}_{x \in \mathcal{A}}$ being disjoint and thus containing each a different rational number, they are at most countably many. This means $\mathcal{A}$ is a countable set. Now (5.1) follows immediately.

This proof, more elegant than our original one, was kindly communicated to us by Omer Adelman. □

Now we prove Corollary 1.6. It follows from Lemma 5.1 that if $(f,g) \in \widetilde{S}_J^{(2)}$, then

$$\int_0^1 (f'(x) + g'(x))^2 \, \mathrm{d}x \leqslant 1, \qquad \int_0^1 (f'(x) - g'(x))^2 \, \mathrm{d}x \leqslant 1,$$

from which (cf. [18])

$$|f(1) + g(1)| \leqslant 1, \qquad |f(1) - g(1)| \leqslant 1$$

showing that a limit point cannot be outside the set given in the Corollary.

To show that every point is a limit point, define

$$f(u) = \frac{x(u - 1 + |x|)}{|x|} \mathbb{1}_{\{1-|x| \leq u \leq 1\}}, \qquad g(u) = \frac{yu}{|y|} \mathbb{1}_{\{0 \leq u \leq |y|\}} + y \mathbb{1}_{\{|y| \leq u \leq 1\}}.$$

It is easy to see that $(f,g) \in \widetilde{S}_J^{(2)}$ and $f(1) = x$, $g(1) = y$. So (x,y) is a limit point.

The proof of Corollary 1.5 is a straightforward modification of the above, so we omit it.□

6 Further consequences: additive functionals

Consider the additive functional

$$A(t) = \int_0^t \psi(W(s)) \, \mathrm{d}s = \int_{\mathbb{R}} \psi(x) L(t,x) \, \mathrm{d}x,$$

where ψ is an integrable function such that $\overline{\psi} := \int_{\mathbb{R}} \psi(x) \, \mathrm{d}x \neq 0$. Then by the ratio ergodic theorem (cf. [15], p. 228)

$$\lim_{t \to \infty} \frac{A(t)}{\overline{\psi} \, L(t)} = 1 \qquad \text{a.s.}$$

Hence, introducing

$$\widetilde{V}_t(x) := \frac{A(xt)}{\overline{\psi}\,\sqrt{2t\log\log t}},$$

Theorem 1.1 implies

Corollary 6.1 *With probability one, the set* $\{(U_t, \widetilde{V}_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\mathcal{S}_J^{(2)} := \left\{ (f,g) : \ f \in \mathcal{S}, g \in \mathcal{S}_M, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \le 1,\ f(x)g'(x) = 0 \text{ a.e.} \right\}.$$

On the other hand, there are additive functionals which can be approximated by the principal value $Y(t)$. Let ψ be a function as above and consider its Hilbert transform:

$$\mathcal{H}(\psi)(x) = \frac{1}{\pi}\,\text{p.v.} \int_{\mathbb{R}} \frac{\psi(y)}{x-y}\,\mathrm{d}y,$$

where p.v. indicates that the integral should be considered as a principal value. It was shown in [11] that if ψ is a Borel function on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^{\kappa} |\psi(x)|\,\mathrm{d}x < \infty,$$

for some $\kappa > 0$, then for all sufficiently small $\varepsilon > 0$, when $t \to \infty$,

$$B(t) := \int_0^t (\mathcal{H}\psi)(W(s))\,\mathrm{d}s = \frac{\overline{\psi}}{\pi}\,Y(t) + o(t^{1/2-\varepsilon}), \qquad \text{a.s.}$$

Introducing the notation

$$\widetilde{Z}_t(x) = \frac{\pi B(xt)}{\overline{\psi}\,\sqrt{8t\log\log t}},$$

we have

Corollary 6.2 *With probability one, the set* $\{(U_t, \widetilde{Z}_t)\}_{t\geqslant 1}$ *is relatively compact in* $\mathcal{C}^{(2)}$*, with limit set equal to*

$$\widetilde{\mathcal{S}}_J^{(2)} = \left\{ (f,g) : \ f \in \mathcal{S}, g \in \mathcal{S}, \int_0^1 (f'(x))^2 + (g'(x))^2)\,\mathrm{d}x \le 1,\ f(x)g'(x) = 0 \text{ a.e.} \right\}.$$

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