On the ranked excursions heights of a Kiefer process

by

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Abstract: Let $(K(s,t), 0 \leq s \leq 1, t \geq 1)$ be a Kiefer process, i.e. a continuous two-parameter centered Gaussian process indexed by $[0,1] \times \mathbb{R}_+$ whose covariance function is given by $\mathbb{E}(K(s_1,t_1)K(s_2,t_2)) = (\min(s_1,s_2) - s_1 s_2)\min(t_1,t_2)$, $0 \leq s_1, s_2 \leq 1$, $t_1, t_2 \geq 0$. For each $t > 0$, the process $K(\cdot,t)$ is a Brownian bridge on the scale of $\sqrt{t}$. Let $M_1^*(t) \geq M_2^*(t) \geq ...M_j^*(t) \geq ...0$ be the ranked excursions heights of $K(\cdot,t)$. In this paper, we study the path properties of the process $t \to M_j^*(t)$. Two laws of iterated logarithm are established to describe the asymptotic behaviors of $M_j^*(t)$ as t goes to infinity.

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1 Introduction

Let $\{B(t),\, t \geq 0\}$ be a standard one-dimensional Brownian motion, i.e. a continuous centered Gaussian process with covariance

$$\mathbb{E}\Big(B(t_1)B(t_2)\Big) = t_1 \wedge t_2, \qquad t_1,\, t_2 \geq 0.$$

It is well-known that almost all sample paths of B consists of countable many zero-free intervals called excursions. Let (a,b) an excursion interval, i.e. $B(a) = B(b) = 0$ and either $B(s) > 0$, $a < s < b$ called positive excursion, or $B(s) < 0$, $a < s < b$ called negative excursion. The height of this excursion is defined by

$$H^* \stackrel{\text{def}}{=} \max_{a<s<b} |B(s)|.$$

Pitman and Yor [11] introduced the ranked heights of excursions up to time t: let

$$H_1(t) \geq H_2(t) \geq \ldots H_j(t) \geq \ldots$$

and

$$H_1^*(t) \geq H_2^*(t) \geq \ldots H_j^*(t) \geq \ldots$$

be the heights of positive and all excursions, resp. of $\{B(s), 0 \leq s \leq t\}$, including the meander heights $\sup_{g_t \leq s \leq t} B(s)$ and $\sup_{g_t \leq s \leq t} |B(s)|$, where g_t denotes the last zero before t.

Let furthermore $\{K(s,t), 0 \leq s \leq 1, t \geq 0\}$ be a Kiefer process, i.e. a continuous two-parameter centered Gaussian process indexed by $[0,1] \times \mathbb{R}_+$ whose covariance function is given by

$$\mathbb{E}\Big(K(s_1,t_1)K(s_2,t_2)\Big) = (\min(s_1,s_2) - s_1 s_2)\, \min(t_1,t_2), \qquad 0 \leq s_1, s_2 \leq 1,\, t_1, t_2 \geq 0.$$

Kiefer [7] introduced this process K to approximate the empirical process. See Csörgő and Révész [4] for detailed studies and related references on Kiefer process and on the invariance principle between empirical process and Kiefer process. Note that for fixed $t > 0$, the process $s \in [0,1] \to \frac{K(s,t)}{\sqrt{t}}$ is a standard Brownian bridge. Denote by

$$M_1(t) \geq M_2(t) \geq ... \geq M_j(t) \geq ...$$

the ranked heights of the positive excursions of the Brownian bridge $K(\cdot,t)$ over the whole time interval $[0,1]$. Denote by

$$M_1^*(t) \geq M_2^*(t) \geq ... \geq M_j^*(t) \geq ...$$

the ranked heights of the excursions of $|K(\cdot,t)|$. By scaling properties, the distributions of $(\frac{M_j(t)}{\sqrt{t}}, j \geq 1)$ and $(\frac{M_j^*(t)}{\sqrt{t}}, j \geq 1)$ are the same as that of the ranked excursions heights of a standard Brownian bridge. See Pitman and Yor [12] for studies on these distribution.

We are interested in the path properties of the processes $t \to M_j(t)$ and $t \to M_j^*(t)$. In particular, we aim at the asymptotic behaviors of $M_j(t)$ and $M_j^*(t)$ as $t \to \infty$.

Observe that $M_1(t) = \sup_{0 \le s \le 1} K(s,t)$ and $M_1^*(t) = \sup_{0 \le s \le 1} |K(s,t)|$. The following LILs are known, see respectively Csörgő and Révész ([4], pp. 81), Mogul'skii [8] and Csáki and Shi [3]:

Theorem A ([4], [8], [3]). *We have*

$$\limsup_{t\to\infty} \frac{M_1^*(t)}{\sqrt{t \log\log t}} = \frac{1}{\sqrt{2}}, \qquad \text{a.s.} \tag{1.1}$$

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} M_1^*(t) = \frac{\pi}{\sqrt{8}}, \qquad \text{a.s.} \tag{1.2}$$

$$\liminf_{t\to\infty} \frac{(\log t)^\chi}{\sqrt{t}} M_1(t) = \begin{cases} 0 & \textit{if } \chi \le \frac{1}{2} \\ \infty & \textit{if } \chi > \frac{1}{2} \end{cases} \qquad \text{a.s.} \tag{1.3}$$

In (1.1), we may replace M_1^* by M_1.

The almost sure behavior of $H_j^*(t)$ was studied in Csáki and Hu [2]:

Theorem B ([2]). *We have*

$$\limsup_{t\to\infty} \frac{H_j^*(t)}{\sqrt{t \log\log t}} = \frac{\sqrt{2}}{2j-1}, \qquad \text{a.s.} \qquad \text{j} \ge 1. \tag{1.4}$$

$$\liminf_{t\to\infty} \frac{(\log t)^\chi}{\sqrt{t}} H_j^*(t) = \begin{cases} 0 & \text{if } \chi \le 1 \\ \infty & \text{if } \chi > 1 \end{cases} \qquad \text{a.s.} \qquad \text{j} \ge 2. \tag{1.5}$$

A natural question is to ask what happens with $(M_j^*(t), t \ge 0)$ for $j \ge 2$. As a process indexed by t, the j-highest heights $M_j^*(t)$ may share some unusual properties different from $M_1^*(t)$. For instance, $t \to M_j^*(t)$ is not continuous for $j \ge 2$ in contrast with the continuity of $M_1^*(\cdot)$.

Theorem 1.1 *Fix $j \ge 1$. We have*

$$\limsup_{t\to\infty} \frac{M_j^*(t)}{\sqrt{t \log\log t}} = \frac{1}{j\sqrt{2}}, \qquad \text{a.s.}$$

The same result remains true when we replace M_j^* by M_j.

It is also of interest to find the liminf behavior of M_j:

Theorem 1.2 *Fix $j \geq 2$. We have*

$$\liminf_{t\to\infty} \frac{(\log t)^\chi}{\sqrt{t}} M_j^*(t) = \begin{cases} 0 & \text{if } \chi \leq \frac{1}{2} \\ \infty & \text{if } \chi > \frac{1}{2} \end{cases} \qquad \text{a.s.}$$

The same result remains true when we replace M_j^* by M_j.

The proof of Theorem 1.1 is based on an estimate on the downcrossings of a Brownian bridge, this estimate will be given in Section 2. To show Theorem 1.2, an usual way is to estimate $\mathbb{P}(\inf_{1\leq t\leq 2} M_j(t) < \epsilon)$ as ϵ goes to 0. This problem remains open to our best knowledge. To overcome this difficulty, we shall adopt the method of Csáki and Shi [3], which consists of reducing the problem for the Kiefer process to that for an Ornstein-Uhlenbeck process. Section 2 also contains several preliminary results to complete the proofs of Theorems 1.1 and 1.2, which will be presented respectively in Sections 3 and 4.

Throughout this paper, $(C_j,\ 1 \leq j \leq 6)$ denote some positive constants whose exact values are unimportant.

2 Downcrossings

Consider a continuous function $f : I = [a, b] \to \mathbb{R}$ with $a, b \in \mathbb{R}$. For two real numbers $x < y$, we define inductively

$$\alpha_1 = \alpha_1(y) \stackrel{\text{def}}{=} \inf\{v \geq a : f(v) \geq y\}, \tag{2.1}$$
$$\beta_k = \beta_k(x) \stackrel{\text{def}}{=} \inf\{v \geq \alpha_k : f(v) \leq x\}, \qquad k \geq 1, \tag{2.2}$$
$$\alpha_k = \alpha_k(y) \stackrel{\text{def}}{=} \inf\{v \geq \beta_{k-1} : f(v) \geq y\}, \qquad k \geq 2, \tag{2.3}$$

with the convention $\inf\emptyset = \infty$. Define the number of downcrossings of (x, y) by f during the time interval I as

$$D_f(x, y; I) = \sup\{k : \alpha_k(y) \leq b\}. \tag{2.4}$$

We adopt the above definition of downcrossings, which is slightly different from the usual one, to keep the following equivalence:

$$\sup_{v\in I} f(v) \geq y \qquad \Longleftrightarrow \qquad D_f(x, y; I) \geq 1.$$

Remark that the condition $\{D_f(x, y; I) \geq 1\}$ does not depend on x. In the following two subsections, we shall discuss respectively the numbers of downcrossings by a standard Brownian motion, a Brownian bridge and by an Ornstein-Uhlenbeck process.

2.1 Brownian bridge

Let $\{B(s),\ s \geq 0\}$ be a standard Brownian motion and let $\{p(s), 0 \leq s \leq 1\}$ be a standard Brownian bridge from 0 to 0. First, we present a preliminary result based on the reflection principle.

Lemma 2.1 *Fix $j \geq 1$ and $\max(x,0) < y$. We have*

$$\mathbb{P}\Big(D_B(x,y;[0,1]) \geq j,\ B(1) \in dz\Big) = \begin{cases} \varphi(2jy - 2(j-1)x - z)dz & \text{if } z \leq y, \\ \varphi(2(j-1)y - 2(j-1)x + z)dz & \text{if } z > y, \end{cases} \tag{2.5}$$

where φ is the standard normal density function.

Proof: We use the reflection principle formulated by (cf., eg. [5])

Fact 2.2 *Let $\{B(s),\ s \geq 0\}$ be a standard Brownian motion and let τ be a stopping time for B. Then*

$$B^{(\tau)}(s) \stackrel{\text{def}}{=} \begin{cases} B(s) & \text{if } 0 \leq s \leq \tau \\ 2B(\tau) - B(s) & \text{if } \tau \leq s \end{cases}$$

is also a standard Brownian motion.

Let us make use of the stopping times $\alpha_k = \alpha_k(y)$ and $\beta_k = \beta_k(x)$ introduced in (2.1)–(2.3), corresponding to $f(t) = B(t)$, $I = [0,1]$.

Our Lemma 2.1 is well-known for $j = 1$.

We illustrate the proof in the simple case $j = 2$, using the reflection principle subsequently for our stopping times. Let $\{B(s),\ 0 \leq s \leq 1\}$ be a Brownian motion such that $\alpha_2 < 1$ and $B(1) = z \leq y$. Then by Fact 2.2, $B_1(s) \stackrel{\text{def}}{=} B^{(\alpha_1)}(s)$, $0 \leq s \leq 1$ is a Brownian motion with $B_1(1) = 2y - z$, β_1 is its first hitting time of $2y - x$ and α_2 is its first hitting time of y after β_1. In the next step consider $B_2(s) \stackrel{\text{def}}{=} B_1^{(\beta_1)}(s)$, $0 \leq s \leq 1$. Then $B_2(1) = 2y - 2x + z$, and α_2 is its first hitting time of $3y - 2x$. Finally, consider $B_3(s) = B_2^{(\alpha_2)}(s)$, $0 \leq s \leq 1$ for which we have $B_3(1) = 4y - 2x - z$. By reversing this procedure, starting from a Brownian motion with endpoints $4y - 2x - z$ at $s = 1$, we get a Brownian motion with $\alpha_1 < 1$ and $B(1) = z$. This proves the first equality of (2.5) in the case $j = 2$. The procedure is similar for $z > y$, except that we stop with B_2, so the last reflection (at α_2) is not performed. Using this idea in obvious manner for the general case $j > 2$, yields our lemma. ■

Since a Brownian bridge $\{p(s),\ 0 \leq s \leq 1\}$ is a Brownian motion conditioned to $p(1) = 0$, we have the following

Corollary 2.3 *For $j \geq 1$ and $\max(x,0) < y$, we have*

$$\mathbb{P}\Big(D_p(x,y;[0,1]) \geq j\Big) = \exp\Big(-2(jy-(j-1)x)^2\Big).$$

Proof: Putting $z = 0$ in (2.5) we get

$$\mathbb{P}\Big(D_p(x,y;[0,1]) \geq j\Big) = \frac{\varphi(2jy-2(j-1)x)}{\varphi(0)} = \exp\Big(-2(jy-(j-1)x)^2\Big).$$

■

Taking $x = 0$, we recover Pitman and Yor [12]'s formula for the distribution of $M_j(1)$:

$$\mathbb{P}\Big(M_j(1) > y\Big) = \mathbb{P}\Big(D_p(0,y;[0,1]) \geq j\Big) = \exp\Big(-2j^2y^2\Big). \tag{2.6}$$

Another corollary can be obtained by taking $x = 0$ and integrating out with respect to z:

Corollary 2.4 *For $j \geq 1$, $y > 0$ we have*

$$\mathbb{P}\Big(H_j(1) > y\Big) = 2\left(1 - \Phi((2j-1)y)\right),$$

where Φ is the standard normal distribution function, and $H_j(1)$ denotes the height of the j-th highest positive Brownian excursion up to time 1.

Now we present an estimate on $\sup_{0\leq t\leq T} M_j^*(t)$.

Proposition 2.5 *Fix $j \geq 2$. There exists some constant $C_1 > 1$ such that for all $u > 0$ and $\lambda \geq \sqrt{u}$, we have*

$$\mathbb{P}\Big(\sup_{0\leq t\leq u} M_j^*(t) > \lambda\Big) \leq C_1\ \exp\left(-2\left(\frac{j\lambda}{\sqrt{u}} - \frac{2j-1}{2}\right)^2\right)$$

Proof: First we prove

Lemma 2.6 *For $0 \leq x < y$, $j \geq 1$, we have*

$$\mathbb{P}\Big(D_{|p|}(x,y;[0,1]) \geq j\Big) \leq 2^j\mathbb{P}\Big(D_p(x,y;[0,1]) \geq j\Big).$$

Proof of Lemma 2.6: Again, we present the proof for $j = 2$. Upcrossings from x to y by $|p|$ are either upcrossings by p from x to y or downcrossings by p from $-x$ to $-y$. Define the following events:

$$\begin{aligned}
A^{++} &\stackrel{\text{def}}{=} \{\text{There are at least two upcrossings by } p \text{ from } x \text{ to } y\} \\
A^{+-} &\stackrel{\text{def}}{=} \{\text{There is at least one downcrossing by } p \text{ from } -x \text{ to } -y \\
&\qquad \text{after an upcrossing by } p \text{ from } x \text{ to } y\} \\
A^{-+} &\stackrel{\text{def}}{=} \{\text{There is at least one upcrossing by } p \text{ from } x \text{ to } y \\
&\qquad \text{after a downcrossing by } p \text{ from } -x \text{ to } -y\} \\
A^{--} &\stackrel{\text{def}}{=} \{\text{There are at least two downcrossings by } p \text{ from } -x \text{ to } -y\}.
\end{aligned}$$

Obviously

$$\mathbb{P}\Big(D_{|p|}(x,y;[0,1]) \geq 2\Big) \leq \mathbb{P}\left(A^{++}\right) + \mathbb{P}\left(A^{+-}\right) + \mathbb{P}\left(A^{-+}\right) + \mathbb{P}\left(A^{--}\right)$$

and by symmetry, $\mathbb{P}(A^{++}) = \mathbb{P}(A^{--})$, $\mathbb{P}(A^{+-}) = \mathbb{P}(A^{-+})$. Moreover, $\mathbb{P}(A^{+-}) \leq \mathbb{P}(A^{++})$, since by Corollary 2.3 we have

$$\mathbb{P}\left(A^{++}\right) = \exp\left(-2(2y-x)^2\right)$$

and an argument, similar to the proof of Lemma 2.1 shows that

$$\mathbb{P}\left(A^{+-}\right) = \exp\left(-8y^2\right).$$

Hence,

$$\mathbb{P}\Big(D_{|p|}(x,y;[0,1]) \geq 2\Big) \leq 4\mathbb{P}\left(A^{++}\right) = 2^2 \exp\left(-2(2y-x)^2\right),$$

proving Lemma 2.6 for $j = 2$. Extension of the above argument in an obvious manner for $j > 2$, proves our Lemma 2.6. ∎

Now we proceed with the proof of Proposition 2.5. For $t > 0$, we define $\sigma_0^{(t)}(0) = 0$ and for $i \geq 1$,

$$\begin{aligned}
\tau_i^{(t)}(x) &\stackrel{\text{def}}{=} \inf\{s \geq \sigma_{i-1}^{(t)}(0) : |K(s,t)| = x\}, \\
\sigma_i^{(t)}(0) &\stackrel{\text{def}}{=} \inf\{s \geq \tau_i^{(t)}(x) : K(s,t) = 0\},
\end{aligned}$$

(write $\tau_i^{(t)}(x) = 1$ if such s does not exist). Therefore,

$$\mathbb{P}\Big(\sup_{0\leq t\leq u} M_j^*(t) > \lambda\Big) = \mathbb{P}\Big(\exists t \in [0,u] : \tau_j^{(t)}(\lambda) < 1\Big) = \mathbb{P}\Big(\Theta \leq u\Big),$$

where we define $\Theta \stackrel{\text{def}}{=} \inf\{t \geq 0 : M_j^*(t) > \lambda\}$. Let $\mathcal{F}_t = \sigma\{K(s,u), 0 \leq s \leq 1, 0 \leq u \leq t\}$. Then Θ is a stopping time with respect to $(\mathcal{F}_t)$. Notice that the process $t \to (K(\cdot,\Theta+t) -$

$K(\cdot,\Theta))$ is independent of $\mathcal{F}_\Theta$ and has same law as $(K(\cdot,t), t\geq 0)$. Using the self similarity: $K(\cdot, v+\Theta) - K(\cdot,\Theta) \stackrel{\text{law}}{=} \sqrt{v}K(\cdot,1)$ for any fixed $v>0$, we get

$$\mathbb{P}\Big(\sup_{0\leq s\leq 1}\Big|K(s,u)-K(s,\Theta)\Big| < \frac{\sqrt{u}}{2}\,\Big|\,\Theta\leq u\Big) \geq \mathbb{P}\Big(\sup_{0\leq s\leq 1}|K(s,1)|<\frac{1}{2}\Big) \stackrel{\text{def}}{=} \frac{2^j}{C_1} > 0.$$

Denote by

$$E_1 \stackrel{\text{def}}{=} \Big\{\sup_{0\leq s\leq 1}\Big|K(s,u)-K(s,\Theta)\Big| < \frac{\sqrt{u}}{2}\Big\} \cap \Big\{\Theta\leq u\Big\},$$

we have shown that

$$\mathbb{P}\Big(\Theta\leq u\Big) \leq 2^{-j}C_1\,\mathbb{P}\Big(E_1\Big).$$

On E_1, we can decompose $K(s,u) = K(s,\Theta) + \widehat{K}(s)$ with $\sup_{0\leq s\leq 1}|\widehat{K}(s)| \leq \frac{\sqrt{u}}{2}$. Since $|K(\tau_i^{(\Theta)}(\lambda),\Theta)| = \lambda$ and $K(\sigma_i^{(\Theta)}(0),\Theta) = 0$ for $1\leq i\leq j$, it follows that for such random times $0 < s_1 \stackrel{\text{def}}{=} \tau_1^{(\Theta)}(\lambda) < v_1 \stackrel{\text{def}}{=} \sigma_1^{(\Theta)}(0) < ... < s_j \stackrel{\text{def}}{=} \tau_j^{(\Theta)}(\lambda) < 1$, we have respectively,

$$|K(s_1,u)| \geq \lambda - \frac{\sqrt{u}}{2}, |K(v_1,u)| \leq \frac{\sqrt{u}}{2}, ..., |K(s_j,u)| \geq \lambda - \frac{\sqrt{u}}{2}.$$

Namely, we have

$$E_1 \subset \Big\{D_{|K(\cdot,u)|}\Big(\frac{\sqrt{u}}{2}, \lambda - \frac{\sqrt{u}}{2}; [0,1]\Big) \geq j\Big\}.$$

It follows from scaling, Corollary 2.3 and Lemma 2.6 that

$$\mathbb{P}\Big(E_1\Big) \leq \mathbb{P}\Big(D_{|K(\cdot,u)|}\Big(\frac{\sqrt{u}}{2}, \lambda - \frac{\sqrt{u}}{2}; [0,1]\Big) \geq j\Big)$$

$$= \mathbb{P}\left(D_{|p|}\left(\frac{1}{2}, \frac{\lambda}{\sqrt{u}} - \frac{1}{2}; [0,1]\right) \geq j\right) \leq 2^j \exp\Big(-2\frac{(j\lambda - (2j-1)\frac{\sqrt{u}}{2})^2}{u}\Big),$$

proving the result. ∎

2.2 Ornstein-Uhlenbeck process

Let us consider a stationary Ornstein-Uhlenbeck process $(U(t), t\geq 0)$ with parameter $\frac{1}{2}$, which is a stationary centered Gaussian process with covariance $\mathbb{E}\Big(U(t)U(s)\Big) = e^{-\frac{|t-s|}{2}}$. We mention a paper by Pitman and Yor [10] for the study of distributions of excursion lengths of U.

Recall some known facts on the hitting times of U. Fix $-\infty \leq z_1 < z_2 \leq \infty$ and define

$$\sigma(z_1, z_2) = \inf\{s \geq 0 : U(s) \notin [z_1, z_2]\}$$

to be the first exit time from the interval $[z_1, z_2]$. Consider the Sturm-Liouville equation:

$$\frac{1}{2}\phi''(x) - \frac{x}{2}\phi'(x) = -\lambda\phi(x), \qquad x \in (z_1, z_2); \qquad \phi(z_i) = 0 \text{ if } |z_i| < \infty,\ i = 1, 2.$$

Fact 2.7 ([14], [6], [9]) *Assume that* $\min(|z_1|, |z_2|) < \infty$. *There is a sequence of simple eigenvalues* $0 < \lambda_1(z_1, z_2) < \ldots < \lambda_n(z_1, z_2) < \ldots$ *whose corresponding eigenfunctions* $\psi_1(z_1, z_2; x), \ldots, \psi_n(z_1, z_2; x), \ldots$ *form a complete orthonormal system with respect to* $m(dx) = e^{-x^2/2}dx$. *The function* $(z_1, z_2) \to \lambda_1(z_1, z_2)$ *is strictly positive and jointly continuous on* $\Xi = \{(z_1, z_2) \in [-\infty, \infty]^2 : z_1 < z_2, \min(|z_1|, |z_2|) < \infty\}$, *strictly increasing in* $z_1 \in (-\infty, z_2]$ *for* $z_2 \leq \infty$ *and strictly decreasing in* $z_2 \in [z_1, \infty)$ *for* $z_1 \geq -\infty$*:*

$$\lambda_1(-\infty, 0) = \lambda_1(0, \infty) = \frac{1}{2}, \qquad \lim_{(z_1,z_2)\to 0} \lambda_1(z_1, z_2) = \infty, \qquad \lim_{(z_1,z_2)\to(-\infty,\infty)} \lambda_1(z_1, z_2) = 0.$$

Fact 2.8 ([14], [6], [9], [1], [3]) *Assume that* $\min(|z_1|, |z_2|) < \infty$. *There exists some constant* $C_2 > 0$ *such that uniformly on* $x \in \mathbb{R}$,

$$\mathbb{P}\Big(\sigma(z_1, z_2) > t \,|\, U(0) = x\Big) = e^{-\lambda_1(z_1,z_2)\,t}\Big(\theta(z_1, z_2)\psi_1(z_1, z_2; x) + r(t, x)\Big),$$

where $\theta(z_1, z_2) = \int_{z_1}^{z_2} \psi_1(z_1, z_2; x)m(dx)$ *and*

$$|r(t, x)| \leq C_2 \exp\Big(\frac{x^2}{9} - \frac{t}{2}\Big).$$

When $z_1 = -z_2 = -z$ *with* $z > 0$, *we get*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\sup_{0\leq s\leq t} |U(s)| < z\Big) = -\lambda_1(-z, z).$$

Moreover, $\lim_{z\to\infty} \lambda_1(-z, z) = 0$.

We shall need the probability that the process U downcrosses a given interval (z_1, z_2) a few times only during $[-t, t]$. This is stated in the following lemma:

Lemma 2.9 *Fix* $-\infty < z_1 < z_2 < \infty$ *and* $k \geq 1$*; We have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(D_U(z_1, z_2; [-t, t]) \leq k\Big) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(D_U(z_1, z_2; [0, 2t]) \leq k\Big) = -2\mu(z_1, z_2),$$

where $\mu(z_1, z_2) \stackrel{\text{def}}{=} \min(\lambda_1(-\infty, z_2), \lambda_1(z_1, \infty)) > 0$. *Moreover, we have*

$$\lim_{z_1,z_2\to 0} \mu(z_1, z_2) = \frac{1}{2}.$$

Proof: The above first equality is due to the stationarity of the Ornstein-Uhlenbeck process. Using again the stopping times α_j and β_j defined in (2.1)–(2.3) associated with $a = 0$, $b = 2t$, $x = z_1$, $y = z_2$, $I = [0, 2t]$ and $f(v) = U(v)$, we have $\mathbb{P}\Big(D_U(z_1, z_2; [0, 2t]) \le k\Big) = \mathbb{P}\Big(\alpha_{k+1} > 2t\Big)$.

Remark that $\alpha_1 = \inf\{s \ge 0 : U(s) \ge z_2\} = \sigma(-\infty, z_2)$. The strong Markov property implies that the random variables of the family $\{\beta_j - \alpha_j, \alpha_{j+1} - \beta_j, \alpha_1, j \ge 1\}$ are mutually independent. Furthermore, $\beta_1 - \alpha_1 = \sigma(z_1, \infty) \circ \theta_{\alpha_1}$ where θ is the usual shift operator. And for $j \ge 2$, $\beta_j - \alpha_j$ (resp: $\alpha_j - \beta_{j-1}$) has the same law as $T_{z_2 \to z_1}$ (resp: $T_{z_1 \to z_2}$), where $T_{x \to y}$ denotes the hitting time of y by an Ornstein-Uhlenbeck process starting from x. Based on Fact 2.8, the simple convolution computation yields that

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}\Big(\alpha_{k+1} > t\Big) \;=\; -\mu(z_1, z_2),$$

and the desired conclusion follows. ∎

2.3 A technical lemma

Recall that $\{p(s), 0 \le s \le 1\}$ denotes a standard Brownian bridge. Let $0 \le y < z/4$ and consider the event

$$G_{y,z} \;=\; \Big\{\exists\, 0 < a_1 < c_1 < b_1 < a_2 < c_2 < b_2 < 1 : \\ |p(a_i)| \le y, |p(b_i)| \le y, |p(c_i)| \ge z, i = 1, 2\Big\} \tag{2.7}$$

Remark that $G_{y,z} \supset G_{0,z}$ and that $G_{0,z}$ is in fact the event that the height of the second highest excursion of $|p(\cdot)|$ is larger than z. We shall need to bound $\mathbb{P}(G_{y,z})$ in the proof of the upper bound of Theorem 1.2.

Lemma 2.10 *There exists an absolute constant $C_3 > 0$ such that for all $0 \le y < \frac{z}{4}$ and $0 < z < \frac{1}{2}$,*

$$\mathbb{P}\Big(G_{y,z}\Big) \le 1 - C_3\, z^2.$$

We can also obtain a lower bound from (2.6) as follows:

$$\mathbb{P}\Big(G_{y,z}\Big) \ge \mathbb{P}\Big(G_{0,z}\Big) = \mathbb{P}\Big(M_2^*(1) \ge z\Big) \ge \mathbb{P}\Big(M_2(1) \ge z\Big) \ge 1 - C_4 z^2,\ \ 0 < z < \frac{1}{2}.$$

Proof of Lemma 2.10: Define

$$T_z^*(p) \;\stackrel{\text{def}}{=}\; \inf\{t \ge 0 : |p(t)| \ge z\}$$
$$\Upsilon(p) \;\stackrel{\text{def}}{=}\; \inf\{t > T_z^*(p) : |p(t)| \le y\},$$

with $\inf \emptyset = \infty$. Let $\delta = z^2$. Observe that

$$G^c_{y,z} \supset \Big\{T^*_z(p) < \delta;\ 1 - 2\delta \le \Upsilon(p) \le 1 - \delta;\ \sup_{\Upsilon \le t \le 1} |p(t)| \le \frac{z}{2}\Big\}.$$

The strong Markov property at $\Upsilon(p)$ implies that

$$\mathbb{P}\Big(G^c_{y,z}\Big) \ge \mathbb{E}\Big(\mathbf{1}_{(T^*_z(p)<\delta;\, 1-2\delta\le\Upsilon(p)\le 1-\delta)}\, f(p(\Upsilon), 1-\Upsilon)\Big),$$

with

$$f(x,s) \stackrel{\text{def}}{=} \mathbb{P}\Big(\text{a Brownian bridge from } x \text{ to } 0 \text{ of length } s \text{ lives in } [-\frac{z}{2}, \frac{z}{2}]\Big).$$

Since $x = y$ or $-y$ and $\delta \le s = 1 - \Upsilon \le 2\delta$, we have from scaling that

$$\begin{aligned}
f(x,s) &= \mathbb{P}\Big(\text{a Brownian bridge from } x \text{ to } 0 \text{ of length } s \text{ lives in } [-\frac{z}{2}, \frac{z}{2}]\Big) \\
&\ge \mathbb{P}\Big(\text{a Brownian bridge from } \tfrac{x}{\sqrt{s}} \text{ to } 0 \text{ of length } 1 \text{ lives in } [-\frac{z}{2\sqrt{s}}, \frac{z}{2\sqrt{s}}]\Big) \\
&\ge \inf_{|a|\le\frac14} \mathbb{P}\Big(\text{a Brownian bridge from } x \text{ to } 0 \text{ of length } 1 \text{ lives in } [-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}]\Big) \\
&\stackrel{\text{def}}{=} C_5 > 0,
\end{aligned}$$

hence we have shown that

$$\mathbb{P}\Big(G^c_{y,z}\Big) \ge C_5 \mathbb{P}\Big(T^*_z(p) < \delta;\ 1 - 2\delta \le \Upsilon(p) \le 1 - \delta\Big).$$

Recall the following absolute continuity between the law of Brownian bridge and that of Brownian motion: Denote by $\mathbb{P}_{0,0}$ the law of $p(\cdot)$ and by $\mathbb{P}_0$ that of $B(\cdot)$, on the canonical space $(X_t, \mathcal{X}_t)_{0\le t\le 1}$, we have for any $t < 1$,

$$d\mathbb{P}_{0,0}\,|_{\mathcal{X}_t} = \frac{1}{\sqrt{1-t}} \exp\Big(-\frac{X_t^2}{2(1-t)}\Big)\, d\mathbb{P}_0\,|_{\mathcal{X}_t}, \qquad t < 1.$$

Applying the above formula to the stopping time Υ and observing that $|X(\Upsilon(X))| = y$, we obtain that

$$\begin{aligned}
&\mathbb{P}\Big(T^*_z(p) < \delta;\ 1 - 2\delta \le \Upsilon(p) \le 1 - \delta\Big) \\
&= \mathbb{E}_0\Big(\mathbf{1}_{(T^*_z(X)<\delta;1-2\delta\le\Upsilon(X)\le1-\delta)}\, \frac{1}{\sqrt{1-\Upsilon(X)}} \exp\Big(-\frac{y^2}{2(1-\Upsilon(X))}\Big)\Big) \\
&\ge \frac{1}{\sqrt{2\delta}} e^{-\frac{y^2}{2\delta}}\, \mathbb{P}_0\Big(T^*_z(X) < \frac{\delta}{2}\Big)\, \mathbb{P}_z\Big(1 - 2\delta \le T_y(X) \le 1 - \frac{3\delta}{2}\Big),
\end{aligned}$$

where $\mathbb{P}_z$ means that the Brownian motion starts from z and $T_y(X)$ denotes the first hitting time at y. Using the well-known distribution of the first hitting time: $\mathbb{P}_z(T_y(X) \in dt) =$

$\frac{z-y}{\sqrt{2\pi t^3}} e^{-(z-y)^2/(2t)} dt$, and the relation: $y \le z/4$, $\delta = z^2$, we obtain that the above probability is bounded below by $C_6 z^2$. Assembling these estimates, we get

$$\mathbb{P}\Big(G^c_{y,z}\Big) \ge C_3 z^2,$$

for some universal constant $C_3 > 0$. ■

3 Proof of Theorem 1.1

We begin with the proof of the upper bound:

$$\limsup_{t\to\infty} \frac{M_j^*(t)}{\sqrt{t \log\log t}} \le \frac{1}{j\sqrt{2}}, \qquad \text{a.s.} \tag{3.1}$$

This follows from Proposition 2.5: Fix an arbitrary constant $a > \frac{1}{j\sqrt{2}}$. Let $n \ge 3$ and $t_n = e^{n/\log n}$. We have from Proposition 2.5 that

$$\mathbb{P}\Big(\sup_{0\le t\le t_{n+1}} M_j^*(t) > a\sqrt{t_n \log\log t_n}\Big) \le C_1 \exp\Big(-(2j^2a^2 + o(1)) \log\log t_n\Big),$$

whose sum over n converges; this in view of a simple application of Borel-Cantelli lemma yields (3.1).

Now, fix an arbitrary constant $a < \frac{1}{j\sqrt{2}}$. It suffices to prove that

$$\limsup_{t\to\infty} \frac{M_j(t)}{\sqrt{t \log\log t}} \ge a, \qquad \text{a.s..} \tag{3.2}$$

To this end, let $t_n = n^n$ and $\lambda_n = a\sqrt{t_n \log\log t_n}$, we consider the event

$$E_n \stackrel{\text{def}}{=} \Big\{M_j(t_n) > \lambda_n\Big\},$$

which is $\mathcal{F}_{t_n} \stackrel{\text{def}}{=} \sigma\{K(s,u), 0 \le s \le 1, 0 \le u \le t_n\}$-measurable. If we can show that

$$\sum_n \mathbb{P}\Big(E_n \,|\, \mathcal{F}_{t_{n-1}}\Big) = \infty, \qquad \text{a.s.} \tag{3.3}$$

then according to Lévy's version of Borel-Cantelli lemma (cf. [13]), we get $\mathbb{P}\Big(E_n, \text{ i.o. }\Big) = 1$ hence (3.2).

Consider the process $\widetilde{K}(s,u) \stackrel{\text{def}}{=} K(s,u+t_{n-1}) - K(s,t_{n-1})$ for $0 \le s \le 1$ and $u \ge 0$. The independent increment property says that $\widetilde{K}(\cdot,\cdot)$ is independent of $\mathcal{F}_{t_{n-1}}$ and has the same law as $K(\cdot,\cdot)$. Fix a small $\epsilon > 0$ such that $2j^2a^2(1+2\epsilon) \le (1-2\epsilon)$.

Recall the notation $D_{\tilde{K}(\cdot,t_n-t_{n-1})}$ in Section 2 for the downcrossings by the process $\widetilde{K}(\cdot,t_n-t_{n-1})$. Observe that

$$\Big\{D_{\tilde{K}(\cdot,t_n-t_{n-1})}(-\epsilon\lambda_n,(1+\epsilon)\lambda_n;[0,1]) \ge j\Big\} \cap \Big\{\widetilde{M_1^*}(t_{n-1}) < \epsilon\lambda_n\Big\} \subset E_n,$$

where $\widetilde{M_1^*}(t_{n-1}) \stackrel{\text{def}}{=} \sup_{0\le s\le 1, 0\le u\le t_{n-1}} |\widetilde{K}(s,u)|$. Therefore, we apply Corollary 2.3 and obtain that for all large n,

$$\begin{aligned}
\mathbb{P}\Big(E_n \,|\, \mathcal{F}_{t_{n-1}}\Big) &\ge \mathbf{1}_{(\widetilde{M_1^*}(t_{n-1})<\epsilon\lambda_n)}\,\mathbb{P}\Big(D_{\tilde{K}(\cdot,t_n-t_{n-1})}(-\epsilon\lambda_n,(1+\epsilon)\lambda_n;[0,1]) \ge j\Big)\\
&= \mathbf{1}_{(\widetilde{M_1^*}(t_{n-1})<\epsilon\lambda_n)}\,\mathbb{P}\Big(D_p(-\epsilon\frac{\lambda_n}{\sqrt{t_n-t_{n-1}}},(1+\epsilon)\frac{\lambda_n}{\sqrt{t_n-t_{n-1}}};[0,1]) \ge j\Big)\\
&\ge \mathbf{1}_{(\widetilde{M_1^*}(t_{n-1})<\epsilon\lambda_n)}\,\exp\Big(-2j^2a^2(1+2\epsilon)\log\log t_n\Big)\\
&\ge \mathbf{1}_{(\widetilde{M_1^*}(t_{n-1})<\epsilon\lambda_n)}\,n^{-(1-\epsilon)},
\end{aligned} \tag{3.4}$$

where the above equality is due to the self-similarity: $\widetilde{K}(\cdot,v) \stackrel{\text{law}}{=} \sqrt{v}\,p(\cdot)$ for any fixed $v>0$, and $p(\cdot)$ is a standard Brownian bridge. Now, we apply (1.1) and obtain that almost surely, $\widetilde{M_1^*}(t_{n-1}) < \epsilon\lambda_n$ for all large n. This together with (3.4) implies (3.3), completing the proof of Theorem 1.1. ∎

4 Proof of Theorem 1.2

4.1 Upper bound

It suffices to show that

$$\liminf_{t\to\infty} \frac{\sqrt{\log t}}{\sqrt{t}}\, M_2^*(t) = 0, \qquad \text{a.s.}$$

According to Lévy's version of Borel-Cantelli's lemma (cf. [13]), the above result follows if we can prove that for any constant $\epsilon > 0$ and for some sequence $(t_n \uparrow \infty)$,

$$\sum_n \mathbb{P}\Big(M_2^*(t_n) < \epsilon\sqrt{\frac{t_n}{\log t_n}} \,|\, \mathcal{F}_{t_{n-1}}\Big) = \infty, \qquad \text{a.s.} \tag{4.1}$$

where $\mathcal{F}_t = \sigma\{K(s,u), 0 \le s \le 1, 0 \le u \le t\}$. Let us consider $t_n = n^{3n}$. By means of (1.1), we have almost surely for all large n,

$$\sup_{0\le s\le 1} |K(s,t_{n-1})| \le \sqrt{t_{n-1}\, \log n} \stackrel{\text{def}}{=} \lambda_n. \tag{4.2}$$

Consider large n. Observe that $\lambda_n \le \frac{1}{4}\epsilon\sqrt{\frac{t_n}{\log t_n}} \stackrel{\text{def}}{=} \frac{x_n}{4}$. By the independent increment property,

$$K(\cdot,t_n) = K(\cdot,t_{n-1}) + \widetilde{K}(\cdot, t_n - t_{n-1}),$$

with $\widetilde{K}$ a Kiefer process independent of $\mathcal{F}_{t_{n-1}}$. The key observation is that

$$\begin{aligned} &\Big\{M_2^*(t_n) \ge x_n\Big\} \cap \Big\{\sup_{0\le s\le 1} |K(s,t_{n-1})| \le \lambda_n\Big\} \\ \subset\ &\Big\{\exists 0 < a_1 < c_1 < b_1 < a_2 < c_2 < b_2 < 1 : |\widetilde{K}(a_i, t_n - t_{n-1})| \le \lambda_n, \\ &\quad |\widetilde{K}(b_i, t_n - t_{n-1})| \le \lambda_n, |\widetilde{K}(c_i, t_n - t_{n-1})| \ge x_n - \lambda_n, i = 1,2\Big\} \stackrel{\text{def}}{=} \widetilde{F}_n, \end{aligned}$$

which implies that

$$\widetilde{F}_n^c \cap \Big\{\sup_{0\le s\le 1} |K(s,t_{n-1})| \le \lambda_n\Big\} \subset \Big\{M_2^*(t_n) < x_n\Big\}.$$

It follows from the independence of $\widetilde{F}_n^c$ and $\mathcal{F}_{t_{n-1}}$ that

$$\begin{aligned} \mathbb{P}\Big(M_2^*(t_n) < x_n \,|\, \mathcal{F}_{t_{n-1}}\Big) &\ge \mathbf{1}_{(\sup_{0\le s\le 1}|K(s,t_{n-1})|\le\lambda_n)}\, \mathbb{P}\Big(\widetilde{F}_n^c\Big) \\ &= \mathbf{1}_{(\sup_{0\le s\le 1}|K(s,t_{n-1})|\le\lambda_n)}\, \mathbb{P}\Big(G_{y,z}^c\Big) \\ &\ge C_3 \mathbf{1}_{(\sup_{0\le s\le 1}|K(s,t_{n-1})|\le\lambda_n)}\, z^2 \\ &\ge C_3\, \frac{\epsilon^2}{4}\, \mathbf{1}_{(\sup_{0\le s\le 1}|K(s,t_{n-1})|\le\lambda_n)}\, \frac{1}{n\log n}, \end{aligned}$$

where the above equality is due to scaling with $y = \frac{\lambda_n}{\sqrt{t_n - t_{n-1}}}$, $z = \frac{x_n - \lambda_n}{\sqrt{t_n - t_{n-1}}}$ and $G_{y,z}^c$ was defined in (2.7) and the second inequality follows from Lemma 2.10. The above lower bound together with (4.2) implies (4.1). ∎

4.2 Lower bound

Fix $j \ge 2$ and $\chi > \frac{1}{2}$. We want to show that almost surely for all large t:

$$M_j(t) > \sqrt{t}\,(\log t)^{-\chi}.$$

Consider the two-parameter Ornstein-Uhlenbeck process $(U(v,t), v \in \mathbb{R}, t \geq 0)$ defined by

$$U\Big(\log\big(\frac{s}{1-s}\big), t\Big) = \frac{K(s,t)}{\sqrt{s(1-s)}}, \qquad 0 < s < 1,\ t \geq 0.$$

Namely, $\{U(v,t), v \in \mathbb{R}, t \geq 0\}$ is a centered Gaussian process with covariance

$$\mathbb{E}\Big(U(v_1,t_1)U(v_2,t_2)\Big) = e^{-\frac{|v_1-v_2|}{2}} \min(t_1,t_2), \qquad v_1, v_2 \in \mathbb{R},\ t_1, t_2 \geq 0.$$

Let $0 < \delta < 1$ be small. First, if there exist some (random) times $\delta \leq u_1 < v_1 < \ldots < u_{j-1} < v_{j-1} < u_j \leq 1-\delta$ such that $U\Big(\log(\frac{u_i}{1-u_i}), t\Big) \geq x$ for $i = 1, \ldots, j$ and $U\Big(\log(\frac{v_i}{1-v_i}), t\Big) = 0$ for $i = 1, \ldots, j-1$, then $K(u_i,t) \geq x\sqrt{\delta(1-\delta)}$ and $K(v_i,t) = 0$; This implies in particular that $M_j(t) \geq x\sqrt{\delta(1-\delta)}$.

If we denote by $D_{U(\cdot,t)}(x,y;[-\log(\frac{1-\delta}{\delta}), \log(\frac{1-\delta}{\delta})])$ the number of downcrossings of (x,y) by $U(\cdot,t)$ during the time interval $[-\log(\frac{1-\delta}{\delta}), \log(\frac{1-\delta}{\delta})]$, then

$$\Big\{D_{U(\cdot,t)}(0,x;[-\log\big(\frac{1-\delta}{\delta}\big), \log\big(\frac{1-\delta}{\delta}\big)]) \geq j\Big\} \subset \Big\{M_j(t) \geq x\sqrt{\delta(1-\delta)}\Big\}.$$

Fix a small constant $c = c(\chi) > 0$ whose value will be determined later. Define $n_k = \exp(\frac{k}{\log k})$ and let $\delta_k = (\log n_k)^{-2\chi}$, $I_k = [-\log(\frac{1-\delta_k}{\delta_k}), \log(\frac{1-\delta_k}{\delta_k})]$, $x_k = c\sqrt{n_{k+1}}$ for $k \geq 3$. Consider the event

$$F_k \stackrel{\text{def}}{=} \Big\{\exists t \in [n_k, n_{k+1}) : D_{U(\cdot,t)}(0,x_k;I_k) \leq j-1\Big\}.$$

If we can show that

$$\sum_k \mathbb{P}\Big(F_k\Big) < \infty, \tag{4.3}$$

then the Borel-Cantelli lemma implies that almost surely for all large k, F_k^c realizes; hence for all large t, we have that $n_k \leq t < n_{k+1}$, and $D_{U(\cdot,t)}(0,x_k;I_k) \geq j$, this implies that $M_j(t) \geq x_k\sqrt{\delta_k(1-\delta_k)} \geq \frac{\sqrt{c}}{2}\sqrt{t}(\log t)^{-\chi}$, proving the convergence part of Theorem 1.2, since $\chi > \frac{1}{2}$ is arbitrary.

To estimate $\mathbb{P}\Big(F_k\Big)$, we consider the stopping time ζ with respect to $\mathcal{F}_t^U = \sigma\{U(x,s), x \in \mathbb{R}, s \leq t\}$

$$\zeta = \inf\{t \geq n_k : D_{U(\cdot,t)}(0,x_k;[-v_k,v_k]) \leq j-1\}.$$

We want to estimate $\mathbb{P}\Big(F_k\Big) = \mathbb{P}\Big(\zeta \leq n_{k+1}\Big)$. Define $\widetilde{U}(v,t) \stackrel{\text{def}}{=} U(v,t+\zeta) - U(v,\zeta)$ for $v \in \mathbb{R}$ and $t \geq 0$. The independent increments property says that $\tilde{U}$ is independent of $\mathcal{F}_\zeta^U$ and has the same law as U. On $\{\zeta \leq n_{k+1}\}$, we have $D_{U(\cdot,\zeta)}(0,x_k;I_k) \leq j-1$; Fix a small constant $\epsilon > 0$. Consider the event

$$G_k \stackrel{\text{def}}{=} \Big\{\sup_{\delta_k \leq s \leq 1-\delta_k} \Big|\widetilde{U}\Big(\log(\frac{1-s}{s}), n_{k+1}-\zeta\Big)\Big| < \epsilon x_k;\ \zeta < n_{k+1}\Big\} \subset F_k.$$

Using the scaling property: $\tilde U(\cdot,t) \overset{\text{law}}{=} \sqrt{t}\tilde U(\cdot,1)$ for any fixed $t>0$, we obtain:

$$\begin{aligned}
\mathbb{P}\Big(G_k\Big) &= \mathbb{E}\Big[\mathbf{1}_{(\zeta<n_{k+1})}\mathbb{P}\Big(\sup_{\delta_k\le s\le 1-\delta_k}\Big|\widetilde U\Big(\log(\frac{1-s}{s}), n_{k+1}-s\Big)\Big| < \epsilon x_k\Big)|_{s=\zeta}\Big] \\
&= \mathbb{E}\Big[\mathbf{1}_{(\zeta<n_{k+1})}\mathbb{P}\Big(\sup_{\delta_k\le s\le 1-\delta_k}\Big|\widetilde U\Big(\log(\frac{1-s}{s}), 1\Big)\Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-s}}\Big)|_{s=\zeta\ge n_k}\Big] \\
&\ge \mathbb{P}\Big(\zeta<n_{k+1}\Big)\,\mathbb{P}\Big(\sup_{\delta_k\le s\le 1-\delta_k}\Big|\widetilde U\Big(\log(\frac{1-s}{s}), 1\Big)\Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}}\Big). \qquad (4.4)
\end{aligned}$$

Observe that on G_k, the number of downcrossings of $(-\epsilon x_k,(1+\epsilon)x_k)$ by $U(\cdot,n_{k+1})$ during $I_k=[-\log(\frac{1-\delta_k}{\delta_k}),\log(\frac{1-\delta_k}{\delta_k})]$ can not be larger or equal to j; otherwise, we would get $D_{U(\cdot,\zeta)}(0,x_k;I_k)\ge j$. In view of this remark, we get

$$\begin{aligned}
\mathbb{P}\Big(F_k\Big) &\le \frac{\mathbb{P}\Big(D_{U(\cdot,n_{k+1})}(-\epsilon x_k,(1+\epsilon)x_k;I_k)\le j-1\Big)}{\mathbb{P}\Big(\sup_{\delta_k\le s\le 1-\delta_k}\Big|\widetilde U\Big(\log(\frac{1-s}{s}),1\Big)\Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}}\Big)} \\
&= \frac{\mathbb{P}\Big(D_{U(\cdot,1)}(-\epsilon c,(1+\epsilon)c;I_k)\le j-1\Big)}{\mathbb{P}\Big(\sup_{\delta_k\le s\le 1-\delta_k}\Big|U\Big(\log(\frac{1-s}{s}),1\Big)\Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}}\Big)}, \qquad (4.5)
\end{aligned}$$

by using the scaling property. Now, we apply Facts 2.8 and 2.7. Since $\frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}}\to\infty$, we have

$$\mathbb{P}\Big(\sup_{\delta_k\le s\le 1-\delta_k}\Big|U\Big(\log(\frac{1-s}{s}),1\Big)\Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}}\Big)\ge \delta_k^{o(1)}, \qquad k\to\infty,$$

and as $k\to\infty$, we have from Lemma 2.9 that

$$\mathbb{P}\Big(D_{U(\cdot,1)}(-\epsilon c,(1+\epsilon)c;I_k)\le j-1\Big)\le \delta_k^{(2\mu(-\epsilon c,(1+\epsilon)c)+o(1))} = (\frac{k}{\log k})^{-4\chi\,(\mu(-\epsilon c,(1+\epsilon)c)+o(1))}.$$

Recall that $\chi>\frac12$. Since $\mu(-\epsilon c,(1+\epsilon)c)\to\frac12$ as $c\to 0$, it follows that we can choose a sufficiently small constant $c=c(\chi)>0$ such that $4\chi\,\mu(-\epsilon c,(1+\epsilon)c)>1$. This in view of (4.5) implies that there exists some constant $a>1$ such that for all large k,

$$\mathbb{P}\Big(F_k\Big)\le k^{-a}$$

proving (4.3), as desired. ■

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