On the ranked excursion heights of a Kiefer process

by

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Abstract: Let $(K(s,t), 0 \le s \le 1, t \ge 1)$ be a Kiefer process, i.e. a continuous two-parameter centered Gaussian process indexed by $[0,1]\times\mathbb{R}_+$ whose covariance function is given by $\mathbb{E}(K(s_1,t_1)K(s_2,t_2)) = (s_1 \wedge s_2 - s_1 s_2)\, t_1 \wedge t_2$, $0 \le s_1, s_2 \le 1$, $t_1, t_2 \ge 0$. For each $t > 0$, the process $K(\cdot, t)$ is a Brownian bridge on the scale of $\sqrt{t}$. Let $M_1^*(t) \ge M_2^*(t) \ge ... M_j^*(t) \ge ... 0$ be the ranked excursion heights of $K(\cdot, t)$. In this paper, we study the path properties of the process $t \to M_j^*(t)$. Two laws of the iterated logarithm are established to describe the asymptotic behaviors of $M_j^*(t)$ as t goes to infinity.

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1 Introduction

Let $\{B(t),\, t \geq 0\}$ be a standard one-dimensional Brownian motion, i.e. a continuous centered Gaussian process with covariance

$$\mathbb{E}\Big(B(t_1)B(t_2)\Big) = t_1 \wedge t_2, \qquad t_1,\, t_2 \geq 0.$$

We consider also standard Brownian bridge $\{p(s),\, 0 \leq s \leq 1\}$, i.e. a centered Gaussian process with covariance

$$\mathbb{E}\left(p(s_1)p(s_2)\right) = s_1 \wedge s_2 - s_1 s_2, \qquad 0 \leq s_1,\, s_2 \leq 1.$$

It is well-known that almost all sample paths of B consists of countable many zero-free intervals called excursions. Let (a, b) an excursion interval, i.e. $B(a) = B(b) = 0$ and either $B(s) > 0$, $a < s < b$ called positive excursion, or $B(s) < 0$, $a < s < b$ called negative excursion. The height of excursion is defined by either

$$H \stackrel{\text{def}}{=} \max_{a \leq s \leq b} B(s)$$

or

$$H^* \stackrel{\text{def}}{=} \max_{a \leq s \leq b} |B(s)|.$$

Clearly, $H > 0$ holds only for positive excursions. Pitman and Yor [11] introduced the ranked heights of excursions up to time t: let

$$H_1(t) \geq H_2(t) \geq \ldots H_j(t) \geq \ldots$$

and

$$H_1^*(t) \geq H_2^*(t) \geq \ldots H_j^*(t) \geq \ldots$$

be the heights of positive and all excursions respectively, of $\{B(s), 0 \leq s \leq t\}$, including the meander heights $\sup_{g_t \leq s \leq t} B(s)$ and $\sup_{g_t \leq s \leq t} |B(s)|$, where g_t denotes the last zero of B before t. The ranked heights of excursions of p can be defined similarly.

Let furthermore $\{K(s,t), 0 \leq s \leq 1, t \geq 0\}$ be a Kiefer process, i.e. a continuous two-parameter centered Gaussian process indexed by $[0,1] \times \mathbb{R}_+$ whose covariance function is given by

$$\mathbb{E}\Big(K(s_1,t_1)K(s_2,t_2)\Big) = (s_1 \wedge s_2 - s_1 s_2)\, t_1 \wedge t_2, \qquad 0 \leq s_1, s_2 \leq 1,\, t_1, t_2 \geq 0.$$

Kiefer [7] introduced this process K to approximate the empirical process. See Csörgő and Révész [4] for detailed studies and related references on Kiefer process and on the invariance principle between empirical process and Kiefer process. Note that for fixed $t > 0$, the process $s \in [0,1] \to \frac{K(s,t)}{\sqrt{t}}$ is a standard Brownian bridge. Denote by

$$M_1(t) \geq M_2(t) \geq ... \geq M_j(t) \geq ...$$

the ranked heights of the positive excursions of the Brownian bridge $K(\cdot, t)$ over the whole time interval $[0,1]$. Denote by

$$M_1^*(t) \geq M_2^*(t) \geq ... \geq M_j^*(t) \geq ...$$

the ranked heights of the excursions of $|K(\cdot,t)|$. By scaling properties, the distributions of $(\frac{M_j(t)}{\sqrt{t}}, j \geq 1)$ and $(\frac{M_j^*(t)}{\sqrt{t}}, j \geq 1)$ are the same as that of the ranked excursion heights of a standard Brownian bridge. See Pitman and Yor [12] for studies on these distribution.

We are interested in the path properties of the processes $t \to M_j(t)$ and $t \to M_j^*(t)$. In particular, we aim at the asymptotic behaviors of $M_j(t)$ and $M_j^*(t)$ as $t \to \infty$.

Observe that $M_1(t) = \sup_{0\leq s\leq 1} K(s,t)$ and $M_1^*(t) = \sup_{0\leq s\leq 1} |K(s,t)|$. The following laws of the iterated logarithm are known, see respectively Csörgő and Révész ([4], pp. 81), Mogul'skii [8] and Csáki and Shi [3]:

Theorem A ([4], [8], [3]). *We have*

$$\limsup_{t\to\infty} \frac{M_1^*(t)}{\sqrt{t \log\log t}} = \frac{1}{\sqrt{2}}, \qquad \text{a.s.} \tag{1.1}$$

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} M_1^*(t) = \frac{\pi}{\sqrt{8}}, \qquad \text{a.s.} \tag{1.2}$$

$$\liminf_{t\to\infty} \frac{(\log t)^\chi}{\sqrt{t}} M_1(t) = \begin{cases} 0 & \text{if } \chi \leq \frac{1}{2} \\ \infty & \text{if } \chi > \frac{1}{2} \end{cases} \qquad \text{a.s.} \tag{1.3}$$

In (1.1) *we may replace* $M_1^*(t)$ *by* $M_1(t)$.

The almost sure behavior of $H_j^*(t)$ was studied in Csáki and Hu [2]:

Theorem B ([2]). *We have*

$$\limsup_{t\to\infty} \frac{H_j^*(t)}{\sqrt{t \log\log t}} = \frac{\sqrt{2}}{2j-1}, \qquad \text{a.s.} \qquad j \geq 1 \tag{1.4}$$

$$\liminf_{t\to\infty} \frac{(\log t)^\chi}{\sqrt{t}} H_j^*(t) = \begin{cases} 0 & \text{if } \chi \leq 1 \\ \infty & \text{if } \chi > 1 \end{cases} \qquad \text{a.s.} \qquad j \geq 2. \tag{1.5}$$

A natural question is to ask what happens with $(M_j^*(t), t \geq 0)$ for $j \geq 2$. As a process indexed by t, the j-highest heights $M_j^*(t)$ may share some unusual properties different from $M_1^*(t)$. For instance, $t \to M_j^*(t)$ is not continuous for $j \geq 2$ in contrast with the continuity of $M_1^*(\cdot)$.

Theorem 1.1 *Fix $j \geq 1$. We have*

$$\limsup_{t\to\infty} \frac{M_j^*(t)}{\sqrt{t \log\log t}} = \frac{1}{j\sqrt{2}}, \qquad \text{a.s.}$$

The same result remains true when $M_j^*(t)$ is replaced by $M_j(t)$.

It is also of interest to find the liminf behavior of $M_j(\cdot)$:

Theorem 1.2 *Fix $j \geq 2$. We have*

$$\liminf_{t\to\infty} \frac{(\log t)^\chi}{\sqrt{t}} M_j^*(t) = \begin{cases} 0 & \text{if } \chi \leq \frac{1}{2} \\ \infty & \text{if } \chi > \frac{1}{2} \end{cases} \qquad \text{a.s.}$$

The same result remains true when we replace $M_j^*(t)$ by $M_j(t)$.

Comparing (1.2) with Theorem 1.2, we can see that the liminf behaviors of M_1^* and M_j^* $(j > 1)$ are completely different.

The proof of Theorem 1.1 is based on an estimate on the downcrossings of a Brownian bridge, this estimate will be given in Section 2. To show Theorem 1.2, a usual way would be to estimate $\mathbb{P}(\inf_{1\leq t\leq 2} M_j(t) < \epsilon)$ as ϵ goes to 0. This problem remains open to our best knowledge. To overcome this difficulty, we shall adopt the method of Csáki and Shi [3], which consists of reducing the problem for the Kiefer process to that for an Ornstein-Uhlenbeck process. Section 2 also contains several preliminary results to complete the proofs of Theorems 1.1 and 1.2, which will be presented respectively in Sections 3 and 4.

Throughout this paper, $(C_k, 1 \leq k \leq 6)$ denote some positive constants whose exact values are unimportant.

2 Downcrossings

Consider a continuous function $f : I = [a, b] \to \mathbb{R}$ with $a, b \in \mathbb{R}$. For two real numbers $x < y$, we define inductively

$$\alpha_1 = \alpha_1(y) \stackrel{\text{def}}{=} \inf\{v \geq a : f(v) \geq y\}, \tag{2.1}$$

$$\beta_k = \beta_k(x) \stackrel{\text{def}}{=} \inf\{v \geq \alpha_k : f(v) \leq x\}, \qquad k \geq 1, \tag{2.2}$$

$$\alpha_k = \alpha_k(y) \stackrel{\text{def}}{=} \inf\{v \geq \beta_{k-1} : f(v) \geq y\}, \qquad k \geq 2, \tag{2.3}$$

with the convention $\inf \emptyset = \infty$. Define the number of downcrossings of (x, y) by f during the time interval I as

$$D_f(x, y; I) = \sup\{k : \alpha_k(y) \leq b\}. \tag{2.4}$$

We adopt the above definition of downcrossings, which is slightly different from the usual one, to keep the following equivalence:

$$\sup_{v\in I} f(v) \geq y \qquad \Longleftrightarrow \qquad D_f(x,y;I) \geq 1.$$

Remark that the condition $\{D_f(x,y;I) \geq 1\}$ does not depend on x. In the following two subsections, we shall discuss respectively the numbers of downcrossings by a standard Brownian motion, a Brownian bridge and by an Ornstein-Uhlenbeck process.

2.1 Brownian motion and Brownian bridge

Let $\{B(s),\ s\geq 0\}$ be a standard Brownian motion and let $\{p(s), 0\leq s\leq 1\}$ be a standard Brownian bridge from 0 to 0. First, we present a preliminary result based on the reflection principle.

Lemma 2.1 *Fix $j\geq 1$ and $\max(x,0)<y$. We have*

$$\mathbb{P}\Big(D_B(x,y;[0,1]) \geq j,\ B(1)\in dz\Big) = \begin{cases} \varphi(2jy-2(j-1)x-z)dz & \text{if } z\leq y, \\ \varphi(2(j-1)y-2(j-1)x+z)dz & \text{if } z>y, \end{cases} \tag{2.5}$$

where φ is the standard normal density function.

Proof: We use the reflection principle formulated by (cf., e.g. [5])

Fact 2.2 *Let $\{B(s),\ s\geq 0\}$ be a standard Brownian motion and let τ be a stopping time for B. Then*

$$B^{(\tau)}(s) \stackrel{\text{def}}{=} \begin{cases} B(s) & \text{if } 0\leq s\leq \tau \\ 2B(\tau)-B(s) & \text{if } \tau\leq s \end{cases}$$

is also a standard Brownian motion.

Let us make use of the stopping times $\alpha_k=\alpha_k(y)$ and $\beta_k=\beta_k(x)$ introduced in (2.1)–(2.3), corresponding to $f(t)=B(t)$, $I=[0,1]$.

Our Lemma 2.1 is well-known for $j=1$.

We illustrate the proof in the simple case $j=2$, using the reflection principle subsequently for our stopping times. Let $\{B(s),\ 0\leq s\leq 1\}$ be a Brownian motion such that $\alpha_2<1$ and $B(1)=z\leq y$. Then by Fact 2.2, $B_1(s)\stackrel{\text{def}}{=} B^{(\alpha_1)}(s)$, $0\leq s\leq 1$ is a Brownian motion with $B_1(1)=2y-z$, β_1 is its first hitting time of $2y-x$ and α_2 is its first hitting time of y after

β_1. In the next step consider $B_2(s) \stackrel{\text{def}}{=} B_1^{(\beta_1)}(s)$, $0 \le s \le 1$. Then $B_2(1) = 2y - 2x + z$, and α_2 is its first hitting time of $3y - 2x$. Finally, consider $B_3(s) = B_2^{(\alpha_2)}(s)$, $0 \le s \le 1$ for which we have $B_3(1) = 4y - 2x - z$. By reversing this procedure, starting from a Brownian motion with endpoints $4y - 2x - z$ at $s = 1$, we get a Brownian motion with $\alpha_1 < 1$ and $B(1) = z$. This proves the first equality of (2.5) in the case $j = 2$. The procedure is similar for $z > y$, except that we stop with B_2, so the last reflection (at α_2) is not performed. Using this idea in obvious manner for the general case $j > 2$, yields our lemma. ∎

Since a Brownian bridge $\{p(s),\ 0 \le s \le 1\}$ is a Brownian motion conditioned to $B(1) = 0$, we have the following

Corollary 2.3 *For $j \ge 1$ and $\max(x, 0) < y$, we have*

$$\mathbb{P}\Big(D_p(x, y; [0,1]) \ge j\Big) = \exp\Big(- 2(jy - (j-1)x)^2\Big).$$

Proof: Putting $z = 0$ in (2.5) we get

$$\mathbb{P}\Big(D_p(x, y; [0,1]) \ge j\Big) = \frac{\varphi(2jy - 2(j-1)x)}{\varphi(0)} = \exp\Big(- 2(jy - (j-1)x)^2\Big).$$

∎

Taking $x = 0$, we recover Pitman and Yor [12]'s formula for the distribution of $M_j(1)$:

$$\mathbb{P}\Big(M_j(1) > y\Big) = \mathbb{P}\Big(D_p(0, y; [0,1]) \ge j\Big) = \exp\Big(- 2j^2y^2\Big). \tag{2.6}$$

Another corollary can be obtained by taking $x = 0$ and integrating with respect to z:

Corollary 2.4 *For $j \ge 1$, $y > 0$ we have*

$$\mathbb{P}\Big(H_j(1) > y\Big) = 2\left(1 - \Phi((2j-1)y)\right),$$

where Φ is the standard normal distribution function, and $H_j(1)$ denotes the height of the j-th highest positive Brownian excursion up to time 1.

Now we present an estimate on $\sup_{0 \le t \le T} M_j^*(t)$.

Proposition 2.5 *Fix $j \ge 2$. There exists some constant $C_1 > 1$ such that for all $u > 0$ and $\lambda \ge \sqrt{u}$, we have*

$$\mathbb{P}\Big(\sup_{0 \le t \le u} M_j^*(t) > \lambda\Big) \le C_1 \exp\left(- 2\left(\frac{j\lambda}{\sqrt{u}} - \frac{2j-1}{2}\right)^2\right)$$

In the proof of Proposition 2.5, we need the following lemma:

Lemma 2.6 *For $0 \le x < y$, $j \ge 1$, we have*

$$\mathbb{P}\Big(D_{|p|}(x,y;[0,1]) \ge j\Big) \le 2^j \mathbb{P}\Big(D_p(x,y;[0,1]) \ge j\Big).$$

Proof of Lemma 2.6: Again, we present the proof for $j = 2$. Upcrossings from x to y by $|p|$ are either upcrossings by p from x to y or downcrossings by p from $-x$ to $-y$. Define the following events:

$$\begin{aligned}
A^{++} &\stackrel{\text{def}}{=} \{\text{There are at least two upcrossings by } p \text{ from } x \text{ to } y\}\\
A^{+-} &\stackrel{\text{def}}{=} \{\text{There is at least one downcrossing by } p \text{ from } -x \text{ to } -y\\
&\qquad \text{after an upcrossing by } p \text{ from } x \text{ to } y\}\\
A^{-+} &\stackrel{\text{def}}{=} \{\text{There is at least one upcrossing by } p \text{ from } x \text{ to } y\\
&\qquad \text{after a downcrossing by } p \text{ from } -x \text{ to } -y\}\\
A^{--} &\stackrel{\text{def}}{=} \{\text{There are at least two downcrossings by } p \text{ from } -x \text{ to } -y\}.
\end{aligned}$$

Obviously

$$\mathbb{P}\Big(D_{|p|}(x,y;[0,1]) \ge 2\Big) \le \mathbb{P}\left(A^{++}\right) + \mathbb{P}\left(A^{+-}\right) + \mathbb{P}\left(A^{-+}\right) + \mathbb{P}\left(A^{--}\right)$$

and by symmetry, $\mathbb{P}(A^{++}) = \mathbb{P}(A^{--})$, $\mathbb{P}(A^{+-}) = \mathbb{P}(A^{-+})$. Moreover, $\mathbb{P}(A^{+-}) \le \mathbb{P}(A^{++})$, since by Corollary 2.3 we have

$$\mathbb{P}\left(A^{++}\right) = \exp\left(-2(2y-x)^2\right)$$

and an argument, similar to the proof of Lemma 2.1 shows that

$$\mathbb{P}\left(A^{+-}\right) = \exp\left(-8y^2\right).$$

Hence,

$$\mathbb{P}\Big(D_{|p|}(x,y;[0,1]) \ge 2\Big) \le 4\mathbb{P}\left(A^{++}\right) = 2^2 \exp\left(-2(2y-x)^2\right),$$

proving Lemma 2.6 for $j = 2$. Extension of the above argument in an obvious manner for $j > 2$, proves our Lemma 2.6. ■

Now we proceed with the proof of Proposition 2.5.

Proof of Proposition 2.5: For $t > 0$, we define $\sigma_0^{(t)}(0) = 0$ and for $i \ge 1$,

$$\begin{aligned}
\tau_i^{(t)}(x) &\stackrel{\text{def}}{=} \inf\{s \ge \sigma_{i-1}^{(t)}(0) : |K(s,t)| = x\},\\
\sigma_i^{(t)}(0) &\stackrel{\text{def}}{=} \inf\{s \ge \tau_i^{(t)}(x) : K(s,t) = 0\},
\end{aligned}$$

(write $\tau_i^{(t)}(x) = 1$ if such s does not exist). Therefore,

$$\mathbb{P}\Big(\sup_{0\le t\le u} M_j^*(t) > \lambda\Big) = \mathbb{P}\Big(\exists t \in [0,u] : \tau_j^{(t)}(\lambda) < 1\Big) = \mathbb{P}\Big(\Theta \le u\Big),$$

where we define $\Theta \stackrel{\text{def}}{=} \inf\{t \ge 0 : M_j^*(t) > \lambda\}$. Let $\mathcal{F}_t \stackrel{\text{def}}{=} \sigma\{K(s,u), 0 \le s \le 1, 0 \le u \le t\}$, the sigma-algebra generated by $\{K(s,u), 0 \le s \le 1, 0 \le u \le t\}$. Then Θ is a stopping time with respect to $(\mathcal{F}_t)$. Notice that the process $t \to (K(\cdot, \Theta + t) - K(\cdot, \Theta))$ is independent of $\mathcal{F}_\Theta$ and has the same law as $(K(\cdot,t), t \ge 0)$. Using the self similarity: $K(\cdot, v + \Theta) - K(\cdot, \Theta) \stackrel{\text{law}}{=} \sqrt{v} K(\cdot, 1)$ for any fixed $v > 0$, we get

$$\mathbb{P}\Big(\sup_{0\le s\le 1} \Big|K(s,u) - K(s,\Theta)\Big| < \frac{\sqrt{u}}{2} \,\Big|\, \Theta \le u\Big) \ge \mathbb{P}\Big(\sup_{0\le s\le 1} |K(s,1)| < \frac{1}{2}\Big) \stackrel{\text{def}}{=} \frac{2^j}{C_1} > 0.$$

Denote by

$$E_1 \stackrel{\text{def}}{=} \Big\{\sup_{0\le s\le 1} \Big|K(s,u) - K(s,\Theta)\Big| < \frac{\sqrt{u}}{2}\Big\} \cap \Big\{\Theta \le u\Big\},$$

we have shown that

$$\mathbb{P}\Big(\Theta \le u\Big) \le 2^{-j} C_1 \, \mathbb{P}\Big(E_1\Big).$$

On E_1, we can decompose $K(s,u) = K(s,\Theta) + \widehat{K}(s)$ with $\sup_{0\le s\le 1} |\widehat{K}(s)| \le \frac{\sqrt{u}}{2}$. Since $|K(\tau_i^{(\Theta)}(\lambda), \Theta)| = \lambda$ and $K(\sigma_i^{(\Theta)}(0), \Theta) = 0$ for $1 \le i \le j$, it follows that for such random times $0 < s_1 \stackrel{\text{def}}{=} \tau_1^{(\Theta)}(\lambda) < v_1 \stackrel{\text{def}}{=} \sigma_1^{(\Theta)}(0) < ... < s_j \stackrel{\text{def}}{=} \tau_j^{(\Theta)}(\lambda) < 1$, we have respectively,

$$|K(s_1,u)| \ge \lambda - \frac{\sqrt{u}}{2}, |K(v_1,u)| \le \frac{\sqrt{u}}{2}, ..., |K(s_j,u)| \ge \lambda - \frac{\sqrt{u}}{2}.$$

Namely, we have

$$E_1 \subset \Big\{D_{|K(\cdot,u)|}\Big(\frac{\sqrt{u}}{2}, \lambda - \frac{\sqrt{u}}{2}; [0,1]\Big) \ge j\Big\}.$$

It follows from scaling, Corollary 2.3 and Lemma 2.6 that

$$\mathbb{P}\Big(E_1\Big) \le \mathbb{P}\Big(D_{|K(\cdot,u)|}\Big(\frac{\sqrt{u}}{2}, \lambda - \frac{\sqrt{u}}{2}; [0,1]\Big) \ge j\Big)$$

$$= \mathbb{P}\left(D_{|p|}\left(\frac{1}{2}, \frac{\lambda}{\sqrt{u}} - \frac{1}{2}; [0,1]\right) \ge j\right) \le 2^j \exp\Big(-2\frac{(j\lambda - (2j-1)\frac{\sqrt{u}}{2})^2}{u}\Big),$$

proving the result. ■

2.2 Ornstein-Uhlenbeck process

Let us consider a stationary Ornstein-Uhlenbeck process $(U(t), t \geq 0)$ with parameter $\frac{1}{2}$, which is a stationary centered Gaussian process with covariance $\mathbb{E}\Big(U(t)U(s)\Big) = e^{-\frac{|t-s|}{2}}$. We mention a paper by Pitman and Yor [10] for the study of distributions of excursion lengths of U.

Recall some known facts on the hitting times of U. Fix $-\infty \leq z_1 < z_2 \leq \infty$ and define

$$\sigma(z_1, z_2) = \inf\{s \geq 0 : U(s) \notin [z_1, z_2]\}$$

to be the first exit time from the interval $[z_1, z_2]$. Consider the Sturm-Liouville equation:

$$\frac{1}{2}\phi''(x) - \frac{x}{2}\phi'(x) = -\lambda\phi(x), \qquad x \in (z_1, z_2); \qquad \phi(z_i) = 0 \text{ if } |z_i| < \infty,\ i = 1, 2.$$

Fact 2.7 ([14], [6], [9]) *Assume that* $\min(|z_1|, |z_2|) < \infty$. *There is a sequence of simple eigenvalues* $0 < \lambda_1(z_1, z_2) < \ldots < \lambda_n(z_1, z_2) < \ldots$ *whose corresponding eigenfunctions* $\psi_1(z_1, z_2; x), \ldots, \psi_n(z_1, z_2; x), \ldots$ *form a complete orthonormal system with respect to* $m(dx) = e^{-x^2/2}dx$. *The function* $(z_1, z_2) \to \lambda_1(z_1, z_2)$ *is strictly positive and jointly continuous on* $\Xi = \{(z_1, z_2) \in [-\infty, \infty]^2 : z_1 < z_2, \min(|z_1|, |z_2|) < \infty\}$, *strictly increasing in* $z_1 \in (-\infty, z_2]$ *for* $z_2 \leq \infty$ *and strictly decreasing in* $z_2 \in [z_1, \infty)$ *for* $z_1 \geq -\infty$:

$$\lambda_1(-\infty, 0) = \lambda_1(0, \infty) = \frac{1}{2}, \qquad \lim_{(z_1, z_2) \to 0} \lambda_1(z_1, z_2) = \infty, \qquad \lim_{(z_1, z_2) \to (-\infty, \infty)} \lambda_1(z_1, z_2) = 0.$$

Fact 2.8 ([14], [6], [9], [1], [3]) *Assume that* $\min(|z_1|, |z_2|) < \infty$. *There exists some constant* $C_2 > 0$ *such that uniformly on* $x \in \mathbb{R}$,

$$\mathbb{P}\Big(\sigma(z_1, z_2) > t \,|\, U(0) = x\Big) = e^{-\lambda_1(z_1, z_2)\, t}\Big(\theta(z_1, z_2)\psi_1(z_1, z_2; x) + r(t, x)\Big),$$

where $\theta(z_1, z_2) = \int_{z_1}^{z_2} \psi_1(z_1, z_2; x) m(dx)$ *and*

$$|r(t, x)| \leq C_2 \exp\Big(\frac{x^2}{9} - \frac{t}{2}\Big).$$

When $z_1 = -z_2 = -z$ *with* $z > 0$, *we get*

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}\Big(\sup_{0 \leq s \leq t} |U(s)| < z\Big) = -\lambda_1(-z, z). \tag{2.7}$$

Moreover, $\lim_{z \to \infty} \lambda_1(-z, z) = 0$.

We shall need the probability that the process U downcrosses a given interval (z_1, z_2) only a few times during $[-t, t]$. This is stated in the following lemma:

Lemma 2.9 *Fix $-\infty < z_1 < z_2 < \infty$ and $k \geq 1$; We have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(D_U(z_1, z_2; [-t, t]) \leq k\Big) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(D_U(z_1, z_2; [0, 2t]) \leq k\Big) = -2\mu(z_1, z_2),$$

where $\mu(z_1, z_2) \stackrel{\text{def}}{=} \min(\lambda_1(-\infty, z_2), \lambda_1(z_1, \infty)) > 0$. Moreover, we have

$$\lim_{z_1, z_2 \to 0} \mu(z_1, z_2) = \frac{1}{2}.$$

Note that the constant k, arbitrary but fixed, does not influence the rate of exponential decay of the two probability terms in the above lemma.

Proof: The above first equality is due to the stationarity of the Ornstein-Uhlenbeck process. Using again the stopping times α_j and β_j defined in (2.1)–(2.3) associated with $a = 0$, $b = 2t$, $x = z_1$, $y = z_2$, $I = [0, 2t]$ and $f(v) = U(v)$, we have $\mathbb{P}\Big(D_U(z_1, z_2; [0, 2t]) \leq k\Big) = \mathbb{P}\Big(\alpha_{k+1} > 2t\Big)$.

Remark that $\alpha_1 = \inf\{s \geq 0 : U(s) \geq z_2\} = \sigma(-\infty, z_2)$. The strong Markov property implies that the random variables of the family $\{\alpha_1, \beta_j - \alpha_j, \alpha_{j+1} - \beta_j, j \geq 1\}$ are mutually independent. Furthermore, $\beta_1 - \alpha_1 = \sigma(z_1, \infty) \circ \theta_{\alpha_1}$ where θ is the usual shift operator. And for $j \geq 2$, $\beta_j - \alpha_j$ (resp: $\alpha_j - \beta_{j-1}$) has the same law as $T_{z_2 \to z_1}$ (resp: $T_{z_1 \to z_2}$), where $T_{x\to y}$ denotes the hitting time of y by an Ornstein-Uhlenbeck process starting from x. Based on Fact 2.8, simple convolution computation yields that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\alpha_{k+1} > t\Big) = -\mu(z_1, z_2),$$

and the desired conclusion follows. ■

2.3 A technical lemma

Recall that $\{p(s), 0 \leq s \leq 1\}$ denotes a standard Brownian bridge. Let $0 \leq y < z/4$ and consider the event

$$G_{y,z} = \Big\{\exists\, 0 < a_1 < c_1 < b_1 < a_2 < c_2 < b_2 < 1 : \\ |p(a_i)| \leq y, |p(b_i)| \leq y, |p(c_i)| \geq z, i = 1, 2\Big\} \tag{2.8}$$

Remark that $G_{y,z} \supset G_{0,z}$ and that $G_{0,z}$ is in fact the event that the height of the second highest excursion of $|p(\cdot)|$ is larger than z. We shall need to bound $\mathbb{P}(G_{y,z})$ in the proof of the upper bound of Theorem 1.2.

Lemma 2.10 *There exists an absolute constant* $C_3 > 0$ *such that for all* $0 \le y < \frac{z}{4}$ *and* $0 < z < \frac{1}{2}$,

$$\mathbb{P}\Big(G_{y,z}\Big) \le 1 - C_3\, z^2.$$

We note that this estimate is nearly sharp, since we can also obtain a lower bound from (2.6) as follows:

$$\mathbb{P}\Big(G_{y,z}\Big) \ge \mathbb{P}\Big(G_{0,z}\Big) = \mathbb{P}\Big(M_2^*(1) \ge z\Big) \ge \mathbb{P}\Big(M_2(1) \ge z\Big) \ge 1 - C_4 z^2,\ 0 < z < \frac{1}{2}.$$

Proof of Lemma 2.10: Fix (y, z) such that $0 \le y < \frac{z}{4}$ and $0 < z < \frac{1}{2}$. Define two stopping times for any continuous process $X(\cdot)$:

$$\begin{aligned} T_z^*(X) &\stackrel{\text{def}}{=} \inf\{t \ge 0 : |X(t)| \ge z\} \\ \Upsilon(X) &\stackrel{\text{def}}{=} \inf\{t > T_z^*(X) : |X(t)| \le y\}, \end{aligned}$$

with $\inf \emptyset = \infty$. Observe that

$$G_{y,z}^c \supset \Big\{T_z^*(p) < z^2;\ 1 - 2z^2 \le \Upsilon(p) \le 1 - z^2;\ \sup_{\Upsilon(p) \le t \le 1} |p(t)| \le \frac{z}{2}\Big\}.$$

Applying the strong Markov property at $\Upsilon(p)$, we deduce from the symmetry that

$$\mathbb{P}\Big(G_{y,z}^c\Big) \ge \mathbb{E}\Big(\mathbf{1}_{(T_z^*(p) < z^2;\, 1-2z^2 \le \Upsilon(p) \le 1-z^2)}\, f(y, 1 - \Upsilon(p); z)\Big),$$

because $p(\Upsilon(p)) = \pm y$ on the event $\{\Upsilon(p) < 1\}$, and where the function f is given by

$$f(y, s; z) \stackrel{\text{def}}{=} \mathbb{P}\Big(\text{The Brownian bridge from } y \text{ to } 0 \text{ of length } s \text{ always lives in } \Big[-\frac{z}{2}, \frac{z}{2}\Big]\Big).$$

It follows from the scaling property that for all $z^2 \le s \le 2z^2$,

$$f(y, s; z) = f\left(\frac{y}{\sqrt{s}}, 1; \frac{z}{\sqrt{s}}\right) \ge \inf_{0 \le a \le \frac{1}{4}} f\left(a, 1, \frac{1}{\sqrt{2}}\right) \stackrel{\text{def}}{=} C_5 > 0,$$

because $a = \frac{y}{\sqrt{s}} \le \frac{1}{4}$ and $\frac{z}{\sqrt{s}} \ge \frac{1}{\sqrt{2}}$. Hence we have shown that

$$\mathbb{P}\Big(G_{y,z}^c\Big) \ge C_5\, \mathbb{P}\Big(T_z^*(p) < z^2;\ 1 - 2z^2 \le \Upsilon(p) \le 1 - z^2\Big).$$

Recall the following absolute continuity between the law of a standard Brownian bridge and that of a standard Brownian motion: Denote by $\mathbb{P}_{0,0}$ the law of $p(\cdot)$ and by $\mathbb{P}_0$ that of $B(\cdot)$, on the canonical space $(\mathcal{C}([0,1] \to \mathbb{R}), (X(t), 0 \le t \le 1), (\mathcal{X}_t)_{0 \le t \le 1})$, we have

$$d\mathbb{P}_{0,0}\,|_{\mathcal{X}_t} = \frac{1}{\sqrt{1-t}} \exp\Big(-\frac{X^2(t)}{2(1-t)}\Big)\, d\mathbb{P}_0\,|_{\mathcal{X}_t}, \qquad t < 1.$$

Applying the above formula to the stopping time $\Upsilon(X)$, we have

$$
\begin{aligned}
& \mathbb{P}\Big(T_z^*(p) < z^2;\, 1-2z^2 \le \Upsilon(p) \le 1-z^2\Big) \\
= \;& \mathbb{E}_0\Big(\mathbf{1}_{(T_z^*(X)<z^2;1-2z^2\le\Upsilon(X)\le 1-z^2)}\,\frac{1}{\sqrt{1-\Upsilon(X)}}\exp\Big(-\frac{y^2}{2(1-\Upsilon(X))}\Big)\Big) \\
\ge \;& \frac{e^{-1/32}}{\sqrt{2}}\,\frac{1}{z}\,\mathbb{P}_0\Big(T_z^*(X) < z^2; 1-2z^2 \le \Upsilon(X) \le 1-z^2\Big) \\
\ge \;& \frac{e^{-1/32}}{\sqrt{2}}\,\frac{1}{z}\,\mathbb{P}_0\Big(T_z^*(X) < \frac{z^2}{2}\Big)\,\mathbb{P}_z\Big(1-2z^2 \le T_y(X) \le 1-\frac{3z^2}{2}\Big),
\end{aligned}
$$

where $\mathbb{P}_z$ means that the Brownian motion $X(\cdot)$ starts from z and $T_y(X)$ denotes the first hitting time at y of X. Thanks to the scaling property, the first probability in the above inequality $\mathbb{P}_0\Big(T_z^*(X) < \frac{z^2}{2}\Big)$ is bounded below by some numerical constant. Using the well-known distribution of the Brownian hitting time: $\mathbb{P}_z(T_y(X) \in dt) = \frac{z-y}{\sqrt{2\pi t^3}} e^{-(z-y)^2/(2t)}dt$, we obtain that

$$
\mathbb{P}_z\Big(1-2z^2 \le T_y(X) \le 1-\frac{3z^2}{2}\Big) \ge C_6\, z^3.
$$

Assembling these estimates, we get

$$
\mathbb{P}\Big(G_{y,z}^c\Big) \ge C_3 z^2,
$$

for some universal constant $C_3 > 0$, as desired. ■

3 Proof of Theorem 1.1

We begin with the proof of the upper bound:

$$
\limsup_{t\to\infty} \frac{M_j^*(t)}{\sqrt{t\log\log t}} \;\le\; \frac{1}{j\sqrt{2}}, \qquad \text{a.s.} \tag{3.1}
$$

This follows from Proposition 2.5: Fix an arbitrary constant $a > \frac{1}{j\sqrt{2}}$. Let $n \ge 3$ and $t_n = e^{n/\log n}$. We have from Proposition 2.5 that

$$
\mathbb{P}\Big(\sup_{0\le t\le t_{n+1}} M_j^*(t) > a\sqrt{t_n\log\log t_n}\Big) \le C_1\,\exp\Big(-(2j^2a^2+o(1))\,\log\log t_n\Big),
$$

whose sum over n converges; this in view of a simple application of Borel-Cantelli lemma yields (3.1).

Now, fix an arbitrary constant $a < \frac{1}{j\sqrt{2}}$. It suffices to prove that

$$
\limsup_{t\to\infty} \frac{M_j(t)}{\sqrt{t\log\log t}} \;\ge\; a, \qquad \text{a.s.} \tag{3.2}
$$

To this end, let $t_n = n^n$ and $\lambda_n = a\sqrt{t_n \log\log t_n}$, we consider the event

$$E_n \overset{\text{def}}{=} \Big\{ M_j(t_n) > \lambda_n \Big\},$$

which is $\mathcal{F}_{t_n} \overset{\text{def}}{=} \sigma\{K(s,u), 0 \le s \le 1, 0 \le u \le t_n\}$-measurable. If we can show that

$$\sum_n \mathbb{P}\Big(E_n \,|\, \mathcal{F}_{t_{n-1}} \Big) = \infty, \qquad \text{a.s.} \tag{3.3}$$

then according to Lévy's version of Borel-Cantelli lemma (cf. [13]), we get $\mathbb{P}\Big(E_n, \text{ i.o. }\Big) = 1$ hence (3.2).

Consider the process $\widetilde{K}(s,u) \overset{\text{def}}{=} K(s, u+t_{n-1}) - K(s, t_{n-1})$ for $0 \le s \le 1$ and $u \ge 0$. The independent increment property says that $\widetilde{K}(\cdot,\cdot)$ is independent of $\mathcal{F}_{t_{n-1}}$ and has the same law as $K(\cdot,\cdot)$. Fix a small $\epsilon > 0$ such that $2j^2a^2(1+2\epsilon) \le (1-2\epsilon)$.

Recall the notation $D_{\tilde{K}(\cdot, t_n - t_{n-1})}$ in Section 2 for the downcrossings by the process $\tilde{K}(\cdot, t_n - t_{n-1})$. Observe that

$$\Big\{ D_{\tilde{K}(\cdot, t_n - t_{n-1})}(-\epsilon\lambda_n, (1+\epsilon)\lambda_n; [0,1]) \ \ge\ j \Big\} \cap \Big\{ \widetilde{M}_1^*(t_{n-1}) < \epsilon\lambda_n \Big\} \ \subset\ E_n,$$

where $\widetilde{M}_1^*(t_{n-1}) \overset{\text{def}}{=} \sup_{0\le s\le 1, 0 \le u \le t_{n-1}} |\widetilde{K}(s,u)|$. Therefore, we apply Corollary 2.3 and obtain that for all large n,

$$\begin{aligned}
\mathbb{P}\Big(E_n \,|\, \mathcal{F}_{t_{n-1}} \Big) &\ge \mathbf{1}_{(\widetilde{M}_1^*(t_{n-1}) < \epsilon\lambda_n)}\, \mathbb{P}\Big(D_{\tilde{K}(\cdot, t_n - t_{n-1})}(-\epsilon\lambda_n, (1+\epsilon)\lambda_n; [0,1]) \ \ge\ j \Big) \\
&= \mathbf{1}_{(\widetilde{M}_1^*(t_{n-1}) < \epsilon\lambda_n)}\, \mathbb{P}\Big(D_p\Big(-\epsilon \frac{\lambda_n}{\sqrt{t_n - t_{n-1}}}, (1+\epsilon)\frac{\lambda_n}{\sqrt{t_n - t_{n-1}}}; [0,1]\Big) \ \ge\ j \Big) \\
&\ge \mathbf{1}_{(\widetilde{M}_1^*(t_{n-1}) < \epsilon\lambda_n)}\, \exp\Big(-2j^2a^2(1+2\epsilon)\log\log t_n \Big) \\
&\ge \mathbf{1}_{(\widetilde{M}_1^*(t_{n-1}) < \epsilon\lambda_n)}\, n^{-(1-\epsilon)},
\end{aligned} \tag{3.4}$$

where the above equality is due to the self-similarity: $\widetilde{K}(\cdot, v) \overset{\text{law}}{=} \sqrt{v}\, p(\cdot)$ for any fixed $v > 0$, and $p(\cdot)$ is a standard Brownian bridge. Now, applying (1.1), we obtain that almost surely, $\widetilde{M}_1^*(t_{n-1}) < \epsilon\lambda_n$ for all large n. This together with (3.4) implies (3.3), completing the proof of Theorem 1.1. ■

4 Proof of Theorem 1.2

4.1 Upper bound

It suffices to show that

$$\liminf_{t\to\infty} \frac{\sqrt{\log t}}{\sqrt{t}} M_2^*(t) = 0, \qquad \text{a.s.}$$

According to Lévy's version of Borel-Cantelli's lemma (cf. [13]), the above result follows if we can prove that for any constant $\epsilon > 0$ and for some sequence $(t_n \uparrow \infty)$,

$$\sum_n \mathbb{P}\Big(M_2^*(t_n) < \epsilon\sqrt{\frac{t_n}{\log t_n}} \,|\, \mathcal{F}_{t_{n-1}} \Big) = \infty, \qquad \text{a.s.} \tag{4.1}$$

where $\mathcal{F}_t = \sigma\{K(s,u), 0 \le s \le 1, 0 \le u \le t\}$. Let us consider $t_n = n^{3n}$. By means of (1.1), we have almost surely for all large n,

$$\sup_{0\le s\le 1} |K(s, t_{n-1})| \le \sqrt{t_{n-1}\, \log n} \stackrel{\text{def}}{=} \lambda_n. \tag{4.2}$$

Consider large n. Observe that $\lambda_n \le \frac{1}{4}\epsilon\sqrt{\frac{t_n}{\log t_n}} \stackrel{\text{def}}{=} \frac{x_n}{4}$. By the independent increment property,

$$K(\cdot, t_n) = K(\cdot, t_{n-1}) + \widetilde{K}(\cdot, t_n - t_{n-1}),$$

with $\widetilde{K}$ a Kiefer process independent of $\mathcal{F}_{t_{n-1}}$. The key observation is that

$$\begin{aligned} &\Big\{ M_2^*(t_n) \ge x_n \Big\} \cap \Big\{ \sup_{0\le s\le 1} |K(s,t_{n-1})| \le \lambda_n \Big\} \\ \subset\ & \Big\{ \exists 0 < a_1 < c_1 < b_1 < a_2 < c_2 < b_2 < 1 : |\widetilde{K}(a_i, t_n - t_{n-1})| \le \lambda_n, \\ & \quad |\widetilde{K}(b_i, t_n - t_{n-1})| \le \lambda_n, |\widetilde{K}(c_i, t_n - t_{n-1})| \ge x_n - \lambda_n, i = 1,2 \Big\} \stackrel{\text{def}}{=} \widetilde{F}_n, \end{aligned}$$

which implies that

$$\widetilde{F}_n^c \cap \Big\{ \sup_{0\le s\le 1} |K(s,t_{n-1})| \le \lambda_n \Big\} \subset \Big\{ M_2^*(t_n) < x_n \Big\}.$$

It follows from the independence of $\widetilde{F}_n^c$ and $\mathcal{F}_{t_{n-1}}$ that

$$\begin{aligned} \mathbb{P}\Big(M_2^*(t_n) < x_n \,|\, \mathcal{F}_{t_{n-1}} \Big) &\ge \mathbf{1}_{(\sup_{0\le s\le 1} |K(s,t_{n-1})| \le \lambda_n)}\, \mathbb{P}\Big(\widetilde{F}_n^c \Big) \\ &= \mathbf{1}_{(\sup_{0\le s\le 1} |K(s,t_{n-1})| \le \lambda_n)}\, \mathbb{P}\Big(G_{y,z}^c \Big) \\ &\ge C_3 \mathbf{1}_{(\sup_{0\le s\le 1} |K(s,t_{n-1})| \le \lambda_n)}\, z^2 \\ &\ge C_3\, \frac{\epsilon^2}{4}\, \mathbf{1}_{(\sup_{0\le s\le 1} |K(s,t_{n-1})| \le \lambda_n)}\, \frac{1}{n\log n}, \end{aligned}$$

where the above equality is due to scaling with $y = \frac{\lambda_n}{\sqrt{t_n - t_{n-1}}}$, $z = \frac{x_n - \lambda_n}{\sqrt{t_n - t_{n-1}}}$, $G_{y,z}^c$ is the complement event of $G_{y,z}$ which was defined in (2.8), and the second inequality follows from Lemma 2.10. The above lower bound together with (4.2) implies (4.1). ∎

4.2 Lower bound

Fix $j \geq 2$ and $\chi > \frac{1}{2}$. We want to show that almost surely for all large t:

$$M_j(t) > \sqrt{t}\,(\log t)^{-\chi}.$$

Consider the two-parameter Ornstein-Uhlenbeck process $(U(v,t), v \in \mathbb{R}, t \geq 0)$ defined by

$$U\Big(\log\left(\frac{s}{1-s}\right), t\Big) = \frac{K(s,t)}{\sqrt{s(1-s)}}, \qquad 0 < s < 1,\ t \geq 0.$$

Namely, $\{U(v,t), v \in \mathbb{R}, t \geq 0\}$ is a centered Gaussian process with covariance

$$\mathbb{E}\Big(U(v_1,t_1)U(v_2,t_2)\Big) = e^{-\frac{|v_1-v_2|}{2}}\, t_1 \wedge t_2, \qquad v_1, v_2 \in \mathbb{R},\ \ t_1, t_2 \geq 0.$$

Let $0 < \delta < 1$ be small. First,if there exist some (random) times $\delta \leq u_1 < v_1 < ... < u_{j-1} < v_{j-1} < u_j \leq 1-\delta$ such that $U\Big(\log(\frac{u_i}{1-u_i}), t\Big) \geq x$ for $i = 1, ..., j$ and $U\Big(\log(\frac{v_i}{1-v_i}), t\Big) = 0$ for $i = 1, ..., j-1$, then $K(u_i, t) \geq x\sqrt{\delta(1-\delta)}$ and $K(v_i, t) = 0$. This implies in particular that $M_j(t) \geq x\sqrt{\delta(1-\delta)}$.

Recall (2.4). If we denote by $D_{U(\cdot,t)}(x, y; [-\log(\frac{1-\delta}{\delta}), \log(\frac{1-\delta}{\delta})])$ the number of downcrossings of (x, y) by $U(\cdot, t)$ during the time interval $[-\log(\frac{1-\delta}{\delta}), \log(\frac{1-\delta}{\delta})]$, then

$$\left\{D_{U(\cdot,t)}\left(0, x; \left[-\log\left(\frac{1-\delta}{\delta}\right), \log\left(\frac{1-\delta}{\delta}\right)\right]\right) \geq j\right\} \subset \left\{M_j(t) \geq x\sqrt{\delta(1-\delta)}\right\}.$$

Fix a small constant $c = c(\chi) > 0$ whose value will be determined later. Define $n_k = \exp(\frac{k}{\log k})$ and let $\delta_k = (\log n_k)^{-2\chi}$, $I_k = [-\log(\frac{1-\delta_k}{\delta_k}), \log(\frac{1-\delta_k}{\delta_k})]$, $x_k = c\sqrt{n_{k+1}}$ for $k \geq 3$. Consider the event

$$F_k \stackrel{\text{def}}{=} \Big\{\exists t \in [n_k, n_{k+1}) : D_{U(\cdot,t)}(0, x_k; I_k) \leq j-1\Big\}.$$

If we can show that

$$\sum_k \mathbb{P}\Big(F_k\Big) < \infty, \tag{4.3}$$

then the Borel-Cantelli lemma implies that almost surely for all large k, F_k^c realizes; hence for all large t, we have that $n_k \leq t < n_{k+1}$, and $D_{U(\cdot,t)}(0, x_k; I_k) \geq j$, which implies that $M_j(t) \geq x_k\sqrt{\delta_k(1-\delta_k)} \geq \frac{c}{2}\sqrt{t}\,(\log t)^{-\chi}$. This yields the convergence part of Theorem 1.2, since $\chi > \frac{1}{2}$ is arbitrary.

To estimate $\mathbb{P}\Big(F_k\Big)$, we consider the following stopping time ζ with respect to $\mathcal{F}_t^U = \sigma\{U(x,s), x \in \mathbb{R}, s \leq t\}$:

$$\zeta = \inf\{t \geq n_k : D_{U(\cdot,t)}(0, x_k; I_k) \leq j-1\}.$$

We want to estimate $\mathbb{P}\Big(F_k\Big) = \mathbb{P}\Big(\zeta < n_{k+1}\Big)$. Define $\widetilde{U}(v,t) \stackrel{\text{def}}{=} U(v, t+\zeta) - U(v,\zeta)$ for $v \in \mathbb{R}$ and $t \geq 0$. The independent increments property says that $\tilde{U}$ is independent of $\mathcal{F}^U_\zeta$ and has the same law as U. On $\{\zeta < n_{k+1}\}$, we have $D_{U(\cdot,\zeta)}(0, x_k; I_k) \leq j-1$; Fix a small constant $\epsilon > 0$. Consider the event

$$G_k \stackrel{\text{def}}{=} \Big\{ \sup_{\delta_k \leq s \leq 1-\delta_k} \Big| \widetilde{U}\Big(\log\left(\frac{1-s}{s}\right), n_{k+1} - \zeta \Big) \Big| < \epsilon x_k;\ \zeta < n_{k+1} \Big\} \subset F_k.$$

Using the scaling property: $\tilde{U}(\cdot, t) \stackrel{\text{law}}{=} \sqrt{t}\widetilde{U}(\cdot, 1)$ for any fixed $t > 0$, we obtain:

$$\begin{aligned}
\mathbb{P}\Big(G_k\Big) &= \int_{[n_k, n_{k+1})} \mathbb{P}(\zeta \in dv)\, \mathbb{P}\Big(\sup_{\delta_k \leq s \leq 1-\delta_k} \Big| \widetilde{U}\Big(\log\left(\frac{1-s}{s}\right), n_{k+1} - v \Big) \Big| < \epsilon x_k \Big) \\
&= \int_{[n_k, n_{k+1})} \mathbb{P}(\zeta \in dv)\, \mathbb{P}\Big(\sup_{\delta_k \leq s \leq 1-\delta_k} \Big| \widetilde{U}\Big(\log\left(\frac{1-s}{s}\right), 1 \Big) \Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1} - v}} \Big) \\
&\geq \mathbb{P}\Big(\zeta < n_{k+1}\Big)\, \mathbb{P}\Big(\sup_{\delta_k \leq s \leq 1-\delta_k} \Big| \widetilde{U}\Big(\log\left(\frac{1-s}{s}\right), 1 \Big) \Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1} - n_k}} \Big). \qquad (4.4)
\end{aligned}$$

Observe that on G_k, the number of downcrossings of $(-\epsilon x_k, (1+\epsilon)x_k)$ by $U(\cdot, n_{k+1})$ during $I_k = [-\log(\frac{1-\delta_k}{\delta_k}), \log(\frac{1-\delta_k}{\delta_k})]$ can not be larger or equal to j; otherwise, we would get $D_{U(\cdot,\zeta)}(0, x_k; I_k) \geq j$. In view of this remark, we get

$$\begin{aligned}
\mathbb{P}\Big(F_k\Big) &\leq \frac{\mathbb{P}\Big(D_{U(\cdot, n_{k+1})}(-\epsilon x_k, (1+\epsilon)x_k; I_k) \leq j-1\Big)}{\mathbb{P}\Big(\sup_{\delta_k \leq s \leq 1-\delta_k} \Big|\widetilde{U}\Big(\log(\frac{1-s}{s}), 1\Big)\Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}}\Big)} \\
&= \frac{\mathbb{P}\Big(D_{U(\cdot, 1)}(-\epsilon c, (1+\epsilon)c; I_k) \leq j-1\Big)}{\mathbb{P}\Big(\sup_{\delta_k \leq s \leq 1-\delta_k} \Big|U\Big(\log(\frac{1-s}{s}), 1\Big)\Big| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}}\Big)}, \qquad (4.5)
\end{aligned}$$

by using the scaling property. We shall bound below the denominator and bound above the numerator in (4.5): the denominator equals

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{-\log((1-\delta_k)/\delta_k) \leq v \leq \log((1-\delta_k)/\delta_k)} |U(v,1)| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}} \Big) \\
&= \mathbb{P}\Big(\sup_{0 \leq v \leq 2\log((1-\delta_k)/\delta_k)} |U(v,1)| < \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}} \Big) \\
&\geq \delta_k^{o(1)}, \qquad k \to \infty, \qquad (4.6)
\end{aligned}$$

where the above equality follows from the stationarity and the above inequality follows from (2.7) in Fact 2.8 with $z = \frac{\epsilon x_k}{\sqrt{n_{k+1}-n_k}} \to \infty$. On the other hand, we have from Lemma 2.9 that

$$\mathbb{P}\Big(D_{U(\cdot,1)}(-\epsilon c, (1+\epsilon)c; I_k) \leq j-1\Big) \leq \delta_k^{\big(2\mu(-\epsilon c, (1+\epsilon)c) + o(1)\big)}$$

$$= \left(\frac{k}{\log k}\right)^{-\left(4\chi\,\mu(-\epsilon c,(1+\epsilon)c)+o(1)\right)}. \tag{4.7}$$

Recall that $\chi > \frac{1}{2}$. Since $\mu(-\epsilon c, (1+\epsilon)c) \to \frac{1}{2}$ as $c \to 0$, we can choose a sufficiently small constant $c = c(\chi) > 0$ such that $4\chi\,\mu(-\epsilon c, (1+\epsilon)c) > 1$. Putting (4.6) and (4.7) into (4.5), we obtain some constant $a > 1$ such that for all large k,

$$\mathbb{P}\Big(F_k\Big) \le k^{-a}$$

proving (4.3), as desired. ■

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