Strong approximations of three-dimensional Wiener sausages

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Abstract. In this paper we prove that the centered three-dimensional Wiener sausage can be strongly approximated by a one-dimensional Brownian motion running at a suitable time clock. The strong approximation gives all possible laws of iterated logarithm as well as the convergence in law in terms of process for the normalized Wiener sausage. The proof relies on Le Gall [10]'s fine L^2-norm estimates between the Wiener sausage and the Brownian intersection local times.

Keywords. Wiener sausage, intersection local times, strong approximation.

AMS Classification 2000. 60F15.

1 Introduction

Let $\{B_t, t \geq 0\}$ be a d-dimensional Brownian motion and let $K \subset \mathbb{R}^d$ be a compact set. The Wiener sausages associated to (B_t) and K are the compact sets

$$S^K(0,t) \stackrel{\text{def}}{=} \bigcup_{0 \leq s \leq t} (B_s + K), \qquad t \geq 0.$$

Denote by $m(dx)$ the Lebesgue measure. The volume of Wiener sausage $m(S^K(0,t))$ is a very rich topic of researches, we only mention some recent references ([3], [7], [8]), see their lists of references for related studies.

We restrict our attention to the three dimensional case ($d = 3$). Denote by $\mathcal{C}(K)$ the Newtonian electrostatic capacity of the compact K, see [6] pp. 250. Concerning the asymptotic of $m(S^K(0,t))$ when $t \to \infty$, Kesten, Spitzer and Whitman ([6] pp. 252, Problem 4), Spitzer [17], and Le Gall [11] proved respectively the following results (1.1), (1.2) and (1.3):

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Theorem A *When $t \to \infty$,*

$$\begin{aligned} \frac{1}{t} m(S^K(0,t)) &\overset{\text{a.s.}}{\longrightarrow} \mathcal{C}(K), && (1.1)\\ \mathbb{E}\Big(m(S^K(0,t))\Big) &= \mathcal{C}(K)t + \frac{4}{(2\pi)^{3/2}}\mathcal{C}(K)^2\sqrt{t} + o(\sqrt{t}), && (1.2)\\ \frac{m(S^K(0,t)) - \mathcal{C}(K)t}{\sqrt{t\log t}} &\overset{(d)}{\longrightarrow} \frac{\mathcal{C}(K)^2}{\pi\sqrt{2}}\mathcal{N}, && (1.3) \end{aligned}$$

where $\mathcal{N}$ denotes a centered real-valued Gaussian variable, of variance 1, and $\mathcal{C}(K)$ denotes Newtonian capacity.

This paper aims at establishing a strong approximation of the process $\{m(S^K(0,t)), t \geq 0\}$:

Theorem 1.1 *Let $l \geq 1$ and $K_1, ..., K_l$ be l compact sets in $\mathbb{R}^3$. There exists a one dimensional Brownian motion $(\beta(t), t \geq 0)$ such that almost surely, for all $1 \leq i \leq l$,*

$$m(S^{K_i}(0,t)) = \mathcal{C}(K_i)t + \frac{\mathcal{C}(K_i)^2}{\pi\sqrt{2}}\beta(t\log t) + o(t^{1/2}(\log t)^{1/4+\delta}),$$

for any positive constant $\delta > 0$.

The above strong approximation is in agreement with Bass and Kumagai [2] where they deal with the range of a three-dimensional (and two-dimensional) random walk. Note also that the Brownian motion β can be chosen simultaneously the same for all the l compact sets $(K_i)_{1\leq i\leq l}$, a fact already pointed out by Le Gall [11].

Theorem 1.1 implies the convergence in terms of process of the normalized Wiener sausage to Brownian motion. The error term $o(t^{1/2}(\log t)^{1/4+\delta})$ being sufficiently small, in addition to weak convergence, we can deduce functional almost sure central limit theorem and laws of the iterated logarithm.

Corollary 1.2 *Let $K \subset \mathbb{R}^3$ be a compact set such that $\mathcal{C}(K) > 0$. Put*

$$M_t(x) = \frac{\pi\sqrt{2}}{\mathcal{C}(K)^2\sqrt{t\log t}}\left(m(S^K(0,xt)) - \mathcal{C}(K)xt\right), \quad 0 \leq x \leq 1.$$

We have

$$\begin{aligned} \mathcal{L}\left(M_t(\cdot)\right) &\overset{w}{\longrightarrow} \mathcal{L}(\beta), \quad t\to\infty\\ \frac{1}{\log T}\int_1^T \frac{1}{t}\delta\left(M_t(\cdot)\right)dt &\overset{w}{\longrightarrow} \mathcal{L}(\beta), \quad \text{a.s.,} \quad T\to\infty\\ \limsup_{t\to\infty}\frac{1}{\sqrt{t\log t\log\log t}}\left(m(S^K(0,t)) - \mathcal{C}(K)t\right) &= \frac{\mathcal{C}(K)^2}{\pi}, \quad \text{a.s.}\\ \liminf_{t\to\infty}\left(\frac{\log\log t}{t\log t}\right)^{1/2}\sup_{0\leq s\leq t}|m(S^K(0,s)) - \mathcal{C}(K)s| &= \frac{\mathcal{C}(K)^2}{4}, \quad \text{a.s.,} \end{aligned}$$

where $\overset{w}{\longrightarrow}$ denotes weak convergence in $C[0,1]$, $\mathcal{L}(\cdot)$ denotes the law of the process in bracket and $\delta(f(x))$ stands for a point mass at $f \in C[0,1]$.

The proof of Theorem 1.1 relies on two steps:

(i) Establish the strong approximation of the Wiener sausage by an intersection local time;

(ii) Use a Tanaka-Rosen-Yor ([14], [20]) type formula and approximate the intersection local time by a Brownian motion.

This paper is organized as follows: In Section 2, we give a general estimate on the rate of growth of a process through its L^p-norms; Section 3 is devoted to the intersection local times, in particular to show the above step (i); the step (ii) is done in Section 4. Finally, we prove Theorem 1.1 in Section 5.

Throughout this paper, $\|\xi\|_p$ denotes the L^p-norm of a random variable ξ, while c_p, c'_p, c''_p denote some positive constants, depending on $p > 0$, whose exact values may vary from one paragraph to another.

2 Rate of growth

We shall estimate the rate of growth of some process through its L^p-norms. The following result on the modulus of continuity, due to Barlow and Yor [1], is obtained as a consequence of Garsia-Rodemich-Rumsey lemma:

Lemma 2.1 ([1], (3.b)) *Let $r > 0$. Assume that $(\xi_t, 0 < t < r)$ is a real-valued continuous process such that for some constants $a > 0, b > 1$ and $\kappa > 0$,*

$$\mathbb{E}\,|\xi_t - \xi_s|^a \le \kappa\,|t-s|^b, \quad \forall\, 0 < s \le t < r.$$

Then for any $0 < \gamma < b-1$, there exists some constant $c(a, b, \gamma) > 0$ such that

$$\mathbb{E}\left(\sup_{0<s\neq t<r} \frac{|\xi_t - \xi_s|^a}{|t-s|^\gamma} \right) \le c(a,b,\gamma)\,\kappa\, r^{b-\gamma}.$$

Proposition 2.2 *Let $(X_t, t > 0)$ be a real-valued process. Let $0 < a \le 1$, $b_2 \ge b_1 \ge 0$ be three constants. Assume that for any $p > 1$, there exists some constant $c_p > 0$ such that*

$$\|X_t\|_p \le c_p\, t^a \,(\log t)^{b_1}, \quad \forall\, t > t_0, \tag{2.1}$$

$$\|X_t - X_s\|_p \le c_p\,(t-s)^a\,(\log t)^{b_2}, \quad t_0 \le s \le t-1, \tag{2.2}$$

$$\left\| \sup_{t \le u \le t+1} |X_u - X_t| \right\|_p \le c_p\, t^{a/2}, \quad \forall\ t > t_0, \tag{2.3}$$

for some constant $t_0 > 1$. Then for any $\epsilon > 0$,

$$X_t = o\left(t^a (\log t)^{b_1+\epsilon} \right), \qquad \text{a.s., } t \to \infty.$$

The power $a/2$ in (2.3) can be replaced by any positive constant smaller than a. Note that in (2.2), we only require for the increments $X_t - X_s$ when $t - s > 1$, the local fluctuations are controlled by (2.3).

Proof: Let $p > 4/a$ and $n > t_0$. By (2.3),

$$\begin{aligned}
\mathbb{P}\Big(\max_{t_0\le j\le n}\sup_{j\le t\le j+1}|X_t - X_j| > n^a\Big) &\le \sum_{j=[t_0]}^{n}\mathbb{P}\Big(\sup_{j\le t\le j+1}|X_t - X_j| > n^a\Big)\\
&\le \sum_{j=[t_0]}^{n} n^{-ap}\, c_p^p\,(1+n)^{ap/2}\\
&\le c'\, n^{-ap/2+1},
\end{aligned}$$

whose sum on n converges. Applying Borel-Cantelli's lemma shows that

$$\max_{0\le j\le n}\sup_{j\le t\le j+1}|X_t - X_j| = O(n^a), \qquad \text{a.s., } n\to\infty.$$

Then it remains to prove that

$$X_n = o(n^a\,(\log n)^{b_1+\epsilon}), \qquad \text{a.s., } n\to\infty.$$

Let $0<\epsilon<a/4$ and fix any constant $\eta\in(0,1/2)$ satisfying $\eta b_2 \le a(1-\eta)/2$. Let $p > 8/(\eta\epsilon)$. Consider large j and define $n_j = [e^{j^\eta}]$. Since $\epsilon>0$ is arbitrary, it suffices to prove that

$$\sum_j \mathbb{P}\Big(\max_{n_j\le k\le n_{j+1}}|X_k| \ge n_j^a\,(\log n_j)^{b_1+\epsilon}\Big) < \infty. \tag{2.4}$$

To this end, we have by triangle inequality

$$\begin{aligned}
&\mathbb{P}\Big(\max_{n_j\le k\le n_{j+1}}|X_k| \ge n_j^a\,(\log n_j)^{b_1+\epsilon}\Big)\\
\le\;& \mathbb{P}\Big(|X_{n_j}| \ge \frac{1}{2}n_j^a\,(\log n_j)^{b_1+\epsilon}\Big) + \mathbb{P}\Big(\max_{n_j\le k\le n_{j+1}}|X_k - X_{n_j}| \ge \frac{1}{2}n_j^a\,(\log n_j)^{b_1+\epsilon}\Big)\\
\stackrel{\text{def}}{=}\;& I_1 + I_2.
\end{aligned}$$

Using the L^p-norm of X_{n_j},

$$\begin{aligned}
I_1 &\le \Big(\frac{1}{2}n_j^a\,(\log n_j)^{b_1+\epsilon}\Big)^{-p} c_p^p\, n_j^{ap}\,(\log n_j)^{pb_1}\\
&\le c_p'\, j^{-p\epsilon\eta} \le j^{-8}.
\end{aligned}$$

We shall apply Lemma 2.1 to estimate I_2. Consider a continuous process ξ such that $\xi_k = X_{k+n_j} - X_{n_j}$ for $0\le k\le n_{j+1}-n_j$, and $\xi_\cdot$ is linear on each interval $[k,k+1]$. From (2.2), we easily deduce that for any $0<u,v<n_{j+1}-n_j$,

$$\mathbb{E}|\xi_u - \xi_v|^p \le \begin{cases} c_p^p\,|v-u|^{ap}\,(\log n_{j+1})^{pb_2}, & |u-v|>2;\\[2ex] c_p'\,|v-u|^{p}\,(\log n_{j+1})^{pb_2}, & |u-v|<1.\end{cases}$$

Since $a \le 1$, there exists some constant $c_{a,p} > 0$ such that for any $0 < u, v < n_{j+1} - n_j$,

$$\mathbb{E}|\xi_u - \xi_v|^p \le c_{a,p} (\log n_{j+1})^{b_2 p} |u - v|^{ap}.$$

Applying Lemma 2.1 with $\gamma = p(a - \epsilon/2) < ap - 1$ gives that for some constant $c(\epsilon, p) > 0$

$$\mathbb{E}\Big(\sup_{0 \le u \ne v < n_{j+1} - n_j} \frac{|\xi_v - \xi_u|^p}{|v - u|^{p(a-\epsilon/2)}} \Big) \le c(\epsilon, p) (n_{j+1} - n_j)^{p\epsilon/2} (\log n_{j+1})^{b_2 p}.$$

It follows from Chebyshev's inequality that

$$\begin{aligned}
I_2 &\le \mathbb{P}\Big(\sup_{0 < k \le n_{j+1} - n_j} \frac{|\xi_k|^p}{k^{p(a-\epsilon/2)}} > 2^{-p} \frac{n_j^{ap}}{(n_{j+1} - n_j)^{p(a-\epsilon/2)}} (\log n_j)^{p(b_1+\epsilon)} \Big) \\
&\le c_{\epsilon,p} 2^p \left(\frac{n_{j+1} - n_j}{n_j} \right)^{ap} (\log n_{j+1})^{pb_2} (\log n_j)^{-p(b_1+\epsilon)} \\
&\le c' \, j^{-(1-\eta)pa - p\eta(b_1+\epsilon) + p\eta b_2} \\
&\le c' \, j^{-ap(1-\eta)/2} \le j^{-2},
\end{aligned}$$

by our choice of p and η. Then the sum of I_1 and I_2 over j converges and it proves (2.4) hence the Proposition. □

When the estimate (2.2) on the increments holds for all $t > s$, we may get rid of the condition on the local fluctuations (2.3), as stated in the following result:

Proposition 2.3 *Let $(X_t, t > 0)$ be a real-valued process. Let $0 < a \le 1$, $b_2 \ge b_1 \ge 0$ be three constants. Assume that for any $p > 1$, there exists some constant $c_p > 0$ such that*

$$\begin{aligned}
\|X_t\|_p &\le c_p \, t^a (\log t)^{b_1}, \quad \forall \, t \ge t_0, \\
\|X_t - X_s\|_p &\le c_p (t - s)^a (\log t)^{b_2}, \quad t_0 \le s \le t,
\end{aligned}$$

for some $t_0 > 1$. Then for any $\epsilon > 0$,

$$X_t = o\left(t^a (\log t)^{b_1+\epsilon} \right), \qquad \text{a.s.}, \ t \to \infty.$$

Proof: The proof goes in the same way as in Proposition 2.2. Let us only mention the main difference. Let ϵ, η, p, n_j be the same as in Proposition 2.2. To control the probability

$$\mathbb{P}\Big(\sup_{n_j \le t \le n_{j+1}} |X_t - X_{n_{j+1}}| \ge \frac{1}{2} n_j^a (\log n_j)^{b_1+\epsilon} \Big),$$

we define $\xi_u := X_{u(n_{j+1} - t_0) + t_0}, 0 \le u \le 1$. Then for any $0 < u, v < 1$,

$$\mathbb{E}|\xi_u - \xi_v|^p \le c_p^p n_{j+1}^{ap} (\log n_{j+1})^{b_2 p} |u - v|^{ap}.$$

Applying Lemma 2.1 with $\gamma = p(a - \epsilon/2) < ap - 1$ gives that for some constant $c(\epsilon, p) > 0$

$$\mathbb{E}\Big(\sup_{0 < s \ne t < n_{j+1}} \frac{|X_t - X_s|^p}{|t - s|^{p(a-\epsilon/2)}} \Big) \le c(\epsilon, p) \, n_{j+1}^{p\epsilon/2} (\log n_{j+1})^{b_2 p}.$$

Thus the proposition follows in the same way. □

3 Intersection local times

Denote by $\{\alpha(x,F), x\in\mathbb{R}^3, F$ bounded measurable $F\subset\mathbb{R}_+^2\}$ the family of intersection local times of B. For any F, almost surely $\alpha(x,F)$ is the density of the measure $\int_F dsdu\delta_{(B_s-B_u)}(dx)$, which means that for any test function $f:\mathbb{R}^3\to\mathbb{R}_+$,

$$\int_{(s,u)\in F} dsdu\, f(B_s-B_u)=\int_{\mathbb{R}^3} dx f(x)\alpha(x,F). \tag{3.1}$$

We refer to Rosen [13], Le Gall [9] for the construction of the family $\{\alpha(x,F)\}$. We shall be particularly interested in two families of subsets F:

$$\begin{aligned}\mathcal{A}_t &= \{(s,u): 0\le s\le u-1\le t-1\}, \qquad t>1.\\ \mathcal{F}_\delta &= \{(s,u): 0\le s\le u-\delta\le 1-\delta\}, \qquad 0<\delta<1.\end{aligned}$$

Let us introduce the following notation: For $0\le a<b$, $S_\epsilon^K(a,b)\overset{\text{def}}{=}\bigcup_{a\le t\le b}(B_t+\epsilon K)$ and $S_\epsilon^K=\bigcup_{0\le s\le 1}(B_s+\epsilon K)$ when there is no risk of confusion. Then we have the following scaling property: For $t>1$,

$$\Big(m(S^K(0,t)),\alpha(0,\mathcal{A}_t)\Big)\overset{\text{law}}{=}\Big(t^{3/2}m(S^K_{1/\sqrt{t}}),t^{1/2}\alpha(0,\mathcal{F}_{1/t})\Big).$$

Write

$$\{\xi\}=\xi-\mathbb{E}(\xi).$$

The following L^p-norm estimate is essentially due to Le Gall who proved the case $p=2$ in [11], Theorem 3.1. His arguments can be adopted to deal with the L^p case.

Proposition 3.1 *Fix a compact set $K\subset\mathbb{R}^3$. For any $p\ge 2$, there exists some constant $c_p>0$ depending on p and K such that for all $0<\epsilon<1/2$,*

$$\left\|\frac{1}{\epsilon^2}\{m(S_\epsilon^K)\}-\mathcal{C}(K)^2\{\alpha(0,\mathcal{F}_{\epsilon^2})\}\right\|_p\le c_p.$$

Consequently, for any $t>1$,

$$\left\|\{m(S^K(0,t))\}-\mathcal{C}(K)^2\{\alpha(0,\mathcal{A}_t)\}\right\|_p\le c_p\sqrt{t}.$$

We shall make use of the following Rosenthal inequality, see Petrov [12], Theorem 2.9:

Lemma 3.2 *For any n centered, independent and identically distributed random variables $X_1,...X_n$ there exists a constant $c_p>0$ depending only on $p\ge 2$ such that*

$$\left\|\sum_{i=1}^n X_i\right\|_p\le\begin{cases} c_p\sqrt{n}\,\|X_1\|_p,\\ c_p\Big(\sqrt{n}\,\|X_1\|_2+n^{1/p}\,\|X_1\|_p\Big).\end{cases} \tag{3.2}$$

Proof of Proposition 3.1: The proof of the case $p > 2$ is almost the same as the case $p = 2$ given by Le Gall ([11], pp. 1006). In fact, let n be an integer such that $1/4 < \epsilon 2^{n/2} \le 1/2$. Then using the notation $S_\epsilon^K(a,b) \stackrel{\text{def}}{=} \bigcup_{a\le t\le b}(B_t + \epsilon K)$, $I_-^{(k,q)} = (\frac{2q-2}{2^k}, \frac{2q-1}{2^k})$ and $I_+^{(k,q)} = (\frac{2q-1}{2^k}, \frac{2q}{2^k})$ for $1 \le q \le 2^{k-1}$,

$$\{m(S_\epsilon^K)\} = \sum_{k=1}^{2^n} \left\{ m\left(S_\epsilon^K\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right)\right\} - \sum_{k=1}^{n}\sum_{q=1}^{2^{k-1}} \left\{ m(S_\epsilon^K(I_-^{(k,q)}) \cap S_\epsilon^K(I_+^{(k,q)}))\right\}.$$

Therefore by the independent increment property of Brownian motion, we can apply (3.2) and get

$$\begin{aligned} \left\| \frac{1}{\epsilon^2}\sum_{k=1}^{2^n} \left\{ m\left(S_\epsilon^K\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)\right)\right\}\right\|_p &\le c_p \epsilon^{-2}\, 2^{n/2} \left\| \left\{ m\left(S_\epsilon^K\left(0, \frac{1}{2^n}\right)\right)\right\}\right\|_p \\ &\le c_p \epsilon^{-2} 2^n \sup_{1/4<r\le 1/2} \left\| \{ S_r^K(0,1)\}\right\|_p \\ &\le c_p', \end{aligned} \tag{3.3}$$

where the second inequality follows from the scaling property: $m(S_\epsilon^K(0, \frac{1}{2^n})) \stackrel{\text{law}}{=} 2^{-3n/2}\, m(S_r^K(0,1))$ with $r = \epsilon 2^{-n/2} \in (\frac{1}{4}, \frac{1}{2}]$. On the other hand,

$$\begin{aligned} &\left\| \frac{1}{\epsilon^2}\sum_{k=1}^{n}\sum_{q=1}^{2^{k-1}} \left\{ m(S_\epsilon^K(I_-^{(k,q)}) \cap S_\epsilon^K(I_+^{(k,q)}))\right\} - \mathcal{C}(K)^2\{\alpha(0, \mathcal{F}_{2^{-n}})\}\right\|_p \\ \le\ & \sum_{k=1}^{n} \left\| \frac{1}{\epsilon^2}\sum_{q=1}^{2^{k-1}} \left\{ m(S_\epsilon^K(I_-^{(k,q)}) \cap S_\epsilon^K(I_+^{(k,q)}))\right\} - \mathcal{C}(K)^2\left\{\alpha(I_-^{(k,q)} \times I_+^{(k,q)})\right\}\right\|_p \\ \le\ & c_p \sum_{k=1}^{n} \left(2^{(k-1)/2}\|\xi(\epsilon,k)\|_2 + 2^{(k-1)/p}\|\xi(\epsilon,k)\|_p\right), \end{aligned}$$

by applying the second inequality in (3.2) and where

$$\xi(\epsilon,k) = \frac{1}{\epsilon^2}\left\{ m(S_\epsilon^K(0,2^{-k}) \cap S_\epsilon^K(2^{-k}, 2^{-k+1}))\right\} - \mathcal{C}(K)^2\left\{\alpha((0,2^{-k}) \times (2^{-k}, 2^{-k+1}))\right\}$$

Le Gall ([11], pp.1006) has already shown that

$$\sum_{k=1}^{n} 2^{(k-1)/2}\|\xi(\epsilon,k)\|_2 = O(1). \tag{3.4}$$

We shall prove that

$$\sum_{k=1}^{n} 2^{(k-1)/p}\|\xi(\epsilon,k)\|_p = O(1),$$

which together with (3.3) and (3.4) implies the Proposition.

To this end, by scaling,

$$\begin{aligned}\sum_{k=1}^n 2^{(k-1)/p}\|\xi(\epsilon,k)\|_p &= 2^{-1/p}\sum_{k=1}^n 2^{-(\frac{1}{2}-\frac{1}{p})k}\left\|r_k^{-2}\Big\{m(S_{r_k}^K\cap\widetilde{S}_{r_k}^K)\Big\}-\mathcal{C}(K)^2\big\{\mathcal{L}_2([0,1]^2)\big\}\right\|_p\\ &\le 2^{-1/p+1}\sum_{k=1}^n 2^{-(\frac{1}{2}-\frac{1}{p})k}\left(\left\|r_k^{-2}m(S_{r_k}^K\cap\widetilde{S}_{r_k}^K)\right\|_p+\mathcal{C}(K)^2\left\|\mathcal{L}_2([0,1]^2)\right\|_p\right),\end{aligned}$$

where $r_k=\epsilon 2^{k/2}<1$, $S_{r_k}^K=S_{r_k}^K(0,1)$ and $\mathcal{L}_2([0,1]^2)$ denotes the intersection local times of two independent Brownian motion B and $\widetilde{B}$ over $[0,1]^2$. Using Le Gall ([10], Corollary 3.2), we have that for any compact K, $\left\|\mathcal{L}_2([0,1]^2)\right\|_p<\infty$ and

$$\sup_{0<r<1}\left\|r^{-2}m(S_r^K\cap\widetilde{S}_r^K)\right\|_p<\infty.$$

It follows that

$$\sum_{k=1}^n 2^{(k-1)/p}\|\xi(\epsilon,k)\|_p\le c_p\sum_{k=1}^n 2^{-(\frac{1}{2}-\frac{1}{p})k}=O(1),$$

completing the proof. □

The main technical estimate in this section is the following L^p-norm:

Lemma 3.3 *Let $p\ge 1$. Denote by D the unit ball of $\mathbb{R}^3$. We have for all $t>1$,*

$$||m(S^D(0,t)\cap\widetilde{S}^D(0,\infty))||_p\le c_p\sqrt{t},$$

where $c_p>0$ only depending on p and $\widetilde{S}^D$ denotes the Wiener sausage associated with another Brownian motion independent of B.

We may choose $c_p=c^p$ in the above lemma for some numerical constant $c>1$.

Proof of Lemma 3.3: By scaling

$$m\Big(S^D(0,t)\cap\widetilde{S}^D(0,\infty)\Big)\overset{\text{law}}{=}\epsilon^{-3}\,m\Big(S_\epsilon^D(0,1)\cap\widetilde{S}_\epsilon^D(0,\infty)\Big),$$

with $\epsilon=t^{-1/2}$. It is equivalent to prove that for any integer $p\ge 1$,

$$\mathbb{E}\Big(m\Big(S_\epsilon^D(0,1)\cap\widetilde{S}_\epsilon^D(0,\infty)\Big)\Big)^p\le c'_p\,\epsilon^{2p},\qquad 0<\epsilon<1. \tag{3.5}$$

Denote by $T_\epsilon^K(x)$ the first entry time into $x-\epsilon K$:

$$T_\epsilon^K(x)=\inf\{t\ge 0: B_t\in x-\epsilon K\},$$

with convention that $\inf\emptyset=\infty$. Define

$$\psi(r)=\frac{1}{r}1_{(r<1)}+e^{-r^2/16},\qquad r>0.$$

Recall the following estimate, see Le Gall [11], Lemma 3.2:

$$\mathbb{P}\Big(T^D_\epsilon(x)\le 1\Big) \;\le\; c\,\psi(|x|), \tag{3.6}$$

$$\mathbb{P}\Big(T^D_\epsilon(x)<\infty\Big) \;=\; \frac{\epsilon}{|x|}\wedge 1. \tag{3.7}$$

Note that

$$\begin{aligned}
\Upsilon_p(\epsilon) &\stackrel{\text{def}}{=} \mathbb{E}\Big(m(S^D_\epsilon(0,1)\cap \widetilde{S}^D_\epsilon(0,\infty))\Big)^p \\
&= \int dx_1...dx_p\,\mathbb{P}\Big(\max_{1\le i\le p} T^D_\epsilon(x_i)\le 1\Big)\,\mathbb{P}\Big(\max_{1\le i\le p} T^D_\epsilon(x_i)<\infty\Big) \\
&= \int_{\exists i\ne j:|x_i-x_j|\le 2\epsilon} + \int_{\forall i\ne j:|x_i-x_j|>2\epsilon}, \qquad (3.8)
\end{aligned}$$

with respect to the obvious density. By symmetry, we have

$$\begin{aligned}
\int_{\exists i\ne j:|x_i-x_j|\le 2\epsilon} &\le \frac{p(p-1)}{2}\int dx_1...dx_{p-1}\int_{|x_p-x_{p-1}|\le 2\epsilon} dx_p \\
&\le c\,(2\epsilon)^3\,\frac{p(p-1)}{2}\int dx_1...dx_{p-1}\,\mathbb{P}\Big(\max_{1\le i\le p-1} T^D_\epsilon(x_i)\le 1\Big)\,\mathbb{P}\Big(\max_{1\le i\le p-1} T^D_\epsilon(x_i)<\infty\Big) \\
&\le 4cp^2\epsilon^3\,\Upsilon_{p-1}(\epsilon). \qquad (3.9)
\end{aligned}$$

Let $\mathcal{S}_p$ be the set of permutations on $[1,...,p]$. By the strong Markov property, we have

$$\begin{aligned}
&\mathbb{P}\Big(\max_{1\le i\le p} T^D_\epsilon(x_i)<\infty) \\
\le\;& \sum_{\sigma\in\mathcal{S}_p}\mathbb{P}\Big(T^D_\epsilon(x_{\sigma(1)})\le ...\le T^D_\epsilon(x_{\sigma(p)})<\infty\Big) \\
\le\;& \sum_{\sigma\in\mathcal{S}_p}\mathbb{P}\Big(T^D_\epsilon(x_{\sigma(1)})\le ...\le T^D_\epsilon(x_{\sigma(p-1)})<\infty\Big)\sup_{|y-x_{\sigma(p-1)}|\le\epsilon}\mathbb{P}_z\Big(T^D_\epsilon(x_{\sigma(p)})<\infty\Big) \\
\le\;& \sum_{\sigma\in\mathcal{S}_p}\prod_{i=1}^p\Big(\frac{\epsilon}{(|x_{\sigma(i)}-x_{\sigma(i-1)}|-\epsilon)^+}\wedge 1\Big),
\end{aligned}$$

by means of (3.7), with convention that $\sigma(0)=0$ and $x_0=0$. Similarly, we get from (3.6) that

$$\mathbb{P}\Big(\max_{1\le i\le p} T^D_\epsilon(x_i)\le 1\Big)\le (c_p)^p\,\epsilon^p\sum_{\sigma\in\mathcal{S}_p}\prod_{i=1}^p\psi(|x_{\sigma(i)}-x_{\sigma(i-1)}|-\epsilon)^+).$$

Plugging these into (3.8), we obtain that

$$\int_{\forall i\ne j:|x_i-x_j|>2\epsilon} dx_1...dx_p\,\mathbb{P}\Big(\max_{1\le i\le p} T^D_\epsilon(x_i)\le 1\Big)\,\mathbb{P}\Big(\max_{1\le i\le p} T^D_\epsilon(x_i)<\infty\Big)\le c^p\,p!\,\epsilon^{2p}\,J^*_p,$$

with

$$J^*_p \;\stackrel{\text{def}}{=}\; \max_{\sigma\in\mathcal{S}_p} J_p(\sigma)\stackrel{\text{def}}{=}\max_{\sigma\in\mathcal{S}_p}\int dx_1...dx_p\prod_{i=1}^p\frac{\psi(|x_{\sigma(i)}-x_{\sigma(i-1)}|)}{|x_i-x_{i-1}|}\Big).$$

We are going to prove that there exists some constant $c > 0$ such that

$$J_p^* \leq c^p \, 2^{3p^2}, \tag{3.10}$$

which in view of (3.8) and (3.9) implies (3.5) and completes the proof.

To show (3.10), we firstly remark that the function $y(\in \mathbb{R}^3) \to \int_{\mathbb{R}^3} \frac{\psi(|x|)}{|x-y|} dx$ is continuous on the whole $\mathbb{R}^3$ and goes to 0 when $|y| \to \infty$, hence

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\psi(|x|)}{|x-y|} dx = c < \infty. \tag{3.11}$$

This implies that for any $y, z \in \mathbb{R}^3$,

$$\begin{aligned}
\int dx \frac{\psi(|x|)\, \psi(|x-y|)}{|x-z|} &= \int_{|x-y| \leq |y|/2} + \int_{|x-y| > |y|/2} \\
&\leq \psi(\frac{|y|}{2}) \int_{|x-y| \leq |y|/2} \frac{\psi(|x-y|)}{|x-z|} dx + \psi(\frac{|y|}{2}) \int_{|x-y|>|y|/2} \frac{\psi(|x|)}{|x-z|} dx \\
&\leq 2\, c\, \psi(\frac{|y|}{2}), \qquad (3.12)
\end{aligned}$$

where in the first inequality, we make use of the monotonicity of $\psi(\cdot)$ and the fact that $|x| \geq |y|/2$ on $\{|x-y| \leq |y|/2\}$. We prove (3.10) by recurrence on p. When $p = 1$, $J_1^* = \int \frac{\psi(|x|)}{|x|} dx < \infty$. Let $p \geq 2$ and consider $\sigma \in \mathcal{S}_p$. If $\sigma(p) = p$, we have

$$\begin{aligned}
J_p(\sigma) &= \int dx_1 ... dx_{p-1} \prod_{i=1}^{p-1} \frac{\psi(|x_{\sigma(i)} - x_{\sigma(i-1)}|)}{|x_i - x_{i-1}|} \Big) \int dx_p \frac{\psi(|x_p - x_{\sigma_{p-1}}|)}{|x_p - x_{\sigma_{p-1}}|} \\
&\leq c\, J_{p-1}(\sigma) \leq c\, J_{p-1}^*,
\end{aligned}$$

by (3.11). If $\sigma(p) \neq p$, we denote by $j = j(\sigma, p) \leq p-1$ such that $\sigma(j) = p$. We have

$$\begin{aligned}
J_p(\sigma) &= \int \frac{dx_1 ... dx_{p-1}}{|x_1| ... |x_{p-1} - x_{p-2}|} \prod_{i=1, i \neq j-1, i \neq j}^{p} \psi(|x_{\sigma(i)} - x_{\sigma(i-1)}|) \int dx_p \frac{\psi(|x_{\sigma_j} - x_{\sigma_{j-1}}|) \psi(|x_{\sigma_{j+1}} - x_{\sigma_j}|)}{|x_p - x_{p-1}|} \\
&\leq 2c \int \frac{dx_1 ... dx_{p-1}}{|x_1| ... |x_{p-1} - x_{p-2}|} \prod_{i=1, i \neq j-1, i \neq j}^{p} \psi(|x_{\sigma(i)} - x_{\sigma(i-1)}|)\ \psi(\frac{|x_{\sigma_{j+1}} - x_{\sigma_{j-1}}|}{2}) \\
&\leq 2c \int dx_1 ... dx_{p-1} \prod_{i=1}^{p-1} \frac{\psi(|x_{\tilde\sigma(i)} - x_{\tilde\sigma(i-1)}|/2)}{|x_i - x_{i-1}|}
\end{aligned}$$

where we used (3.12) for the first inequality and the monotonicity of ψ for the second inequality, the new permutation $\tilde\sigma \in \mathcal{S}_{p-1}$ is defined by: $\tilde\sigma(k) = \sigma(k)$ for $k < j$ and $\tilde\sigma(k) = \sigma(k+1)$ for $j \leq k \leq p-1$. By the change of variable $x = 2\tilde x$, we get

$$J_p(\sigma) \leq 2c\, 2^{3(p-1)}\, J_{p-1}(\tilde\sigma) \leq 2^{3p-2}\, c\, J_{p-1}^*.$$

Hence $J_p^* \le 2^{3p-2}\, c\, J_{p-1}^* \le 2^{3p^2} c^p$, yielding that for all $p \ge 1$ and $\epsilon < 1/2$,

$$\mathbb{E}\Big(m\Big(S_\epsilon^D(0,1) \cap \widetilde{S}_\epsilon^D(0,\infty)\Big)^p\Big) \le \epsilon^{2p}\, c\, 2^{4p^2}.$$

□

Corollary 3.4 *Fix the compact set K. For any $p > 1$, there exists some constant $c_p > 0$ such that*

$$\begin{aligned} \big\|\{m(S^K(0,t))\} - \{m(S^K(0,s))\}\big\|_p &\le c_p \sqrt{(t-s)\log t},\ 0 < s < t-1; & (3.13) \\ \Big\|\sup_{t \le u \le t+1} |m(S^K(0,u)) - m(S^K(0,t))|\Big\|_p &\le c_p. & (3.14) \end{aligned}$$

Proof: Consider firstly the case $t - s > 1$. Write $F\backslash G := F \cap G^c$. Define $\widetilde{B}_u := B_{u+s} - B_s, u \ge 0$ and $\widetilde{S}^K$ the associated Wiener sausage. We have

$$\begin{aligned} \{m(S^K(0,t))\} - \{m(S^K(0,s))\} &= \{m(S^K(s,t)\backslash S^K(0,s))\} \\ &\overset{\text{law}}{=} \Big\{m(\widetilde{S}^K(0,t-s)\backslash S^K(0,s))\Big\} \\ &= \Big\{m(\widetilde{S}^K(0,t-s))\Big\} - \Big\{m(\widetilde{S}^K(0,t-s) \cap S^K(0,s))\Big\}. \end{aligned}$$

Firstly, the convergence in law of (1.3) holds in fact in L^p for any $p > 1$. This together with (1.2) shows that

$$\Big\|\Big\{m(\widetilde{S}^K(0,t-s))\Big\}\Big\|_p \le c_p \sqrt{(t-s)\log(3+t-s)}, \qquad \forall\, 0 < s < t-1.$$

Secondly, we have

$$\begin{aligned} \Big\|\Big\{m(\widetilde{S}^K(0,t-s) \cap S^K(0,s))\Big\}\Big\|_p &\le 2\Big\|m(\widetilde{S}^K(0,t-s) \cap S^K(0,s))\Big\|_p \\ &\le 2\Big\|m(\widetilde{S}^K(0,t-s) \cap S^K(0,\infty))\Big\|_p \\ &\le c'_p \sqrt{t-s}, \end{aligned}$$

by Lemma 3.3. Then (3.13) is proven.

For $t \le u \le t+1$, we have

$$\begin{aligned} \Big\|\sup_{t \le u \le t+1} |m(S^K(0,u)) - m(S^K(0,t))|\Big\|_p &= \big\|\{m(S^K(t,u)\backslash S^K(0,t))\}\big\|_p \\ &\le 2\big\|m(S^K(t,t+1)\backslash S^K(0,t))\big\|_p \\ &= 2\Big\|m(\widetilde{S}^K(0,1)\backslash S^K(0,t))\Big\|_p \\ &\le 2\big\|m(S^K(0,1))\big\|_p \\ &\le c_p, \end{aligned}$$

which completes the proof. □

4 Increments of intersection local times

We quote a Tanaka-Rosen-Yor type formula for the intersection local time (see [16], [19], [20]):

Lemma 4.1 ([20], (3.c)) *For any $t \geq 1$, we have*

$$\alpha(0, \mathcal{A}_t) = -\frac{1}{2\pi} \int_1^t dB_u \int_0^{u-1} ds \frac{B_u - B_s}{|B_u - B_s|^3} - \frac{1}{2\pi} \int_1^{t-1} ds \left(\frac{1}{|B_t - B_s|} - \frac{1}{|B_{1+s} - B_s|} \right).$$

We shall estimate the L^p norm of several quantities related to the intersection local times $\alpha(0, \mathcal{A}_t)$. Let us collect some preliminary results in the following lemma:

Lemma 4.2 *Let $t \geq 27$ and $p \geq 2$. There exists some constant $c_p > 0$,*

$$\left\| \int_1^t ds \frac{B_s}{|B_s|^3} \right\|_p \leq c_p \sqrt{\log t} \tag{4.1}$$

$$\left\| \int_1^t ds \frac{B_s}{|B_s|^3} \right\|_2 = \sqrt{2 \log t} + O(1), \tag{4.2}$$

$$\left\| \int_0^{t-1} \frac{ds}{|B_t - B_s|} \right\|_p \leq c_p \sqrt{t}, \tag{4.3}$$

$$\left\| \left\{ \int_0^t \frac{ds}{|B_{1+s} - B_s|} \right\} \right\|_p \leq c_p \sqrt{t}. \tag{4.4}$$

Proof: We shall constantly use the Burkholder-Davis-Gundy inequality for martingales: For any $p > 0$, there exist some constants $c_p > 0$ and $c'_p > 0$ such that for any continuous martingale (M_t), we have

$$c_p \left(\|\langle M \rangle_t\|_{p/2} \right)^{1/2} \leq \left\| \sup_{0 \leq s \leq t} |M_s| \right\|_p \leq c'_p \left(\|\langle M \rangle_t\|_{p/2} \right)^{1/2}. \tag{4.5}$$

We define $R_t \stackrel{\text{def}}{=} |B_t|, t \geq 0$ which is the radial part of the three dimensional Brownian motion B. It is well-known ([15], Chap. 12) that R is a three-dimensional Bessel process and admits the following canonical decomposition:

$$R_t = \gamma_t + \frac{1}{2} \int_0^t \frac{ds}{R_s}, \qquad t \geq 0,$$

for some one-dimensional Brownian motion γ.

Applying Itô's formula to $\frac{x}{|x|}$ gives that

$$\begin{aligned} \int_1^t ds \frac{B_s}{|B_s|^3} &= -\left(\frac{B_t}{|B_t|} - \frac{B_1}{|B_1|} \right) + \int_1^t \frac{dB_s}{|B_s|} - \int_1^t \frac{B_s}{|B_s|} \frac{B_s \circ dB_s}{|B_s|^2} \\ &\stackrel{\text{def}}{=} -\left(\frac{B_t}{|B_t|} - \frac{B_1}{|B_1|} \right) + M_t, \qquad u > 1. \end{aligned} \tag{4.6}$$

Using the Burkholder-Davis-Gundy inequality (4.5) for the martingale $M = (M_u^{(i)})$, $1 \le i \le 3$ implies that

$$\begin{aligned}
\|M_t\|_p &\le c_p \sum_{i=1}^{3} \left\| \langle M^{(i)} \rangle_t \right\|_{p/2}^{1/2} \\
&= c_p \sum_{i=1}^{3} \left\| \int_1^t ds \left(\frac{1}{|B_s|^2} - \frac{(B_s^{(i)})^2}{|B_s|^4} \right) \right\|_{p/2}^{1/2} \\
&\le 6c_p \left\| \int_1^t \frac{ds}{R_s^2} \right\|_{p/2}^{1/2} \\
&\le c'_p \sqrt{\log t},
\end{aligned}$$

where the last inequality is a consequence of the fact that

$$\frac{1}{\log t} \int_1^t \frac{ds}{R_s^2} \to 1, \qquad \text{a.s. and in } L^p. \tag{4.7}$$

Thus (4.1) is proven. Observe that

$$\mathbb{E}|M_t|^2 = \sum_{i=1}^{3} \mathbb{E}\langle M^{(i)} \rangle_t = \sum_{i=1}^{3} \mathbb{E} \int_1^t ds \left(\frac{1}{|B_s|^2} - \frac{(B_s^{(i)})^2}{|B_s|^4} \right) = 2 \log t, \quad t > 1,$$

which together with (4.6) yields (4.2). It follows from Brownian time-reversal that

$$\begin{aligned}
\mathbb{E} \left(\int_0^t \frac{ds}{|B_t - B_s|} \right)^p &= \mathbb{E} \left(\int_0^t \frac{ds}{R_s} \right)^p \\
&= 2^p \mathbb{E} \left(R_t - \gamma_t \right)^p \\
&\le 4^p \left(\mathbb{E}(R_t)^p + \mathbb{E}|\gamma_t|^p \right) \\
&= c'_p t^{p/2},
\end{aligned}$$

by the scaling property. This yields (4.3). To obtain (4.4), we write

$$\int_0^t \frac{ds}{|B_{1+s} - B_s|} = \sum_{j=0}^{[t/2]} \int_{2j}^{2j+1} \frac{ds}{|B_{1+s} - B_s|} + \sum_{j=0}^{[t/2]} \int_{2j+1}^{2j+2} \frac{ds}{|B_{1+s} - B_s|} - \int_t^{2[t/2]+2} \frac{ds}{|B_{1+s} - B_s|}.$$

In each of two sums on j, we have independent and identically distributed variables with common law that of $\int_0^1 \frac{ds}{|B_{1+s}-B_s|} \in L^p$; hence by Rosenthal's inequality (3.2), the two sums centered by their expectations have a L^p norm bounded by $O(\sqrt{t})$. The L^p-norm of the third term itself is less than $\left\| \int_t^{t+2} \frac{ds}{|B_{1+s}-B_s|} \right\|_p = \left\| \int_0^2 \frac{ds}{|B_{1+s}-B_s|} \right\|_p$ is bounded on t. Then (4.4) is proven and completes the proof of Lemma 4.2. □

Lemma 4.3 *For any $\epsilon > 0$, almost surely,*

$$\left\{ \int_1^{t-1} \frac{ds}{|B_{1+s} - B_s|} \right\} = o\left(\sqrt{t}\, (\log t)^\delta \right), \qquad \text{a.s., } t \to \infty, \tag{4.8}$$

$$\int_1^{t-1} \frac{ds}{|B_t - B_s|} = o\left(\sqrt{t}\, (\log t)^\delta \right), \qquad \text{a.s., } t \to \infty, \tag{4.9}$$

Proof: The proof is based on an application of Propositions 2.2 and 2.3. We shall prove that for any $t > s > 2$,

$$\left\| \int_0^{t-1} du \frac{1}{|B_t - B_u|} - \int_0^{s-1} du \frac{1}{|B_s - B_u|} \right\|_p \le c_p \sqrt{(t-s)\log t}. \tag{4.10}$$

Therefore, by virtue of (4.3) and (4.10), Proposition 2.3 implies (4.8). The (4.9) can be proven similarly.

To show (4.10), we have

$$\begin{aligned} & \left\| \int_0^{t-1} du \frac{1}{|B_t - B_u|} - \int_0^{s-1} du \frac{1}{|B_s - B_u|} \right\|_p \\ \le & \left\| \int_{s-1}^{t-1} du \frac{1}{|B_t - B_u|} \right\|_p + \left\| \int_0^{s-1} du \left(\frac{1}{|B_t - B_u|} - \frac{1}{|B_s - B_u|} \right) \right\|_p \\ \le & \; c_p \sqrt{t-s} + \left\| \int_s^t dB_r \int_0^{s-1} du \frac{B_r - B_u}{|B_r - B_u|^3} \right\|_p, \end{aligned}$$

by using respectively (4.3) to obtain the first term and the Itô's formula for the second term. Now, by using (4.5), the L^p norm in the above inequality is smaller than

$$c_p \left\| \int_s^t dr \left| \int_0^{s-1} du \frac{B_r - B_u}{|B_r - B_u|^3} \right|^2 \right\|_{p/2}^{1/2} \le c_p \left(\int_s^t dr \left\| \int_1^r \frac{du}{R_u^2} \right\|_p \right)^{1/2} \le c_p' \sqrt{(t-s)\log t}.$$

□

We can now prove our main estimate in this section:

Lemma 4.4 *For $1 \le s < t$ and $t > 2$,*

$$\begin{aligned} \left\| \int_1^t du \left| \int_0^{u-1} ds \frac{B_u - B_s}{|B_u - B_s|^3} \right|^2 - 2t \log t \right\|_p &\le c_p \, t \sqrt{\log t \, \log\log t}, & (4.11) \\ \left\| \int_s^t du \left| \int_0^{u-1} dv \frac{B_u - B_v}{|B_u - B_v|^3} \right|^2 \right\|_p &\le c_p (t-s) \log t, & (4.12) \end{aligned}$$

Proof: Notice that by Brownian time-reversal and (4.1),

$$\left\| \int_0^{u-1} ds \frac{B_u - B_s}{|B_u - B_s|^3} \right\|_p = \left\| \int_1^u ds \frac{B_s}{|B_s|^3} \right\|_p \le c_p \sqrt{\log u} \qquad u \ge 2, \tag{4.13}$$

which yields (4.12). In view of (4.2), (4.11) is equivalent to prove that

$$\left\| \int_1^t du \left\{ \left| \int_0^{u-1} ds \frac{B_u - B_s}{|B_u - B_s|^3} \right|^2 \right\} \right\|_p \le c_p t \sqrt{\log t \log\log t}.$$

Define

$$Y_u(a,b) = \int_a^b ds \frac{B_u - B_s}{|B_u - B_s|^3}, \qquad 0 < a < b < u.$$

Then by scaling property, $\left\| \int_1^t du \Big\{ \Big| \int_0^{u-1} ds \frac{B_u - B_s}{|B_u - B_s|^3} \Big|^2 \Big\} \right\|_p = t \left\| \int_{\frac{1}{t}}^1 du \Big\{ |Y_u(0, u - \frac{1}{t})|^2 \Big\} \right\|_p$. Then it suffices to show that for $0 < \epsilon < 1/27$,

$$\left\| \int_\epsilon^1 du \Big\{ |Y_u(0, u-\epsilon)|^2 \Big\} \right\|_p \le c_p \sqrt{\log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}}. \tag{4.14}$$

To this end, we remark that for any $0 < a < b < u$,

$$\begin{aligned} \|Y_u(a,b)\|_p &= \left\| \int_{u-b}^{u-a} ds \frac{B_s}{|B_s|^3} \right\|_p \\ &\le 2 + \|M_{u-a} - M_{u-b}\|_p \\ &\le 2 + c_p \left\| \int_{u-b}^{u-a} \frac{ds}{R_s^2} \right\|_{p/2}^{1/2} \\ &\le 2 + c_p \sqrt{\log \frac{u-a}{u-b}}, \end{aligned}$$

by means of (4.7). Choose $\delta = \frac{1}{n} \sim (\log \frac{1}{\epsilon})^{-\theta}$ for some constant $\theta > 2$. We have

$$\int_\epsilon^1 du \Big\{ |Y_u(0,u-\epsilon)|^2 \Big\} = \int_\epsilon^\delta du \Big\{ |Y_u(0,u-\epsilon)|^2 \Big\} + \int_\delta^1 du \Big\{ |Y_u(0,u-\epsilon)|^2 \Big\}.$$

Observe that

$$\begin{aligned} \left\| \int_\epsilon^\delta du \Big\{ |Y_u(0,u-\epsilon)|^2 \Big\} \right\|_p &\le 2 \left\| \int_\epsilon^\delta du |Y_u(0,u-\epsilon)|^2 \right\|_p \\ &\le 2 \int_\epsilon^\delta du \left\| |Y_u(0,u-\epsilon)|^2 \right\|_p \\ &\le c_p \int_\epsilon^\delta du (1 + \log(u/\epsilon)) \\ &\le c_p \delta \log(1/\epsilon) = c_p (\log(1/\epsilon))^{1-\theta}. \end{aligned} \tag{4.15}$$

We claim that

$$\left\| \int_\delta^1 du \Big\{ |Y_u(0,u-\epsilon)|^2 \Big\} - \int_\delta^1 du \Big\{ |Y_u(u-\delta,u-\epsilon)|^2 \Big\} \right\|_p \le c_p \sqrt{\log \frac{1}{\epsilon} \log\log \frac{1}{\epsilon}}. \tag{4.16}$$

In fact, by Cauchy-Schwarz' inequality,

$$\left\| \int_0^1 ds f(u) g(u) \right\|_p \le \left(\left\| \int_0^1 du |f(u)|^2 \right\|_p \right)^{1/2} \left(\left\| \int_0^1 du |g(u)|^2 \right\|_p \right)^{1/2}.$$

Then by writing $Y_u(0,u-\epsilon) = Y_u(0,u-\delta) + Y_u(u-\delta,u-\epsilon)$, we have

$$
\begin{aligned}
&\Big\| \int_\delta^1 du \Big\{ |Y_u(0,u-\epsilon)|^2 \Big\} - \int_\delta^1 du \Big\{ |Y_u(u-\delta,u-\epsilon)|^2 \Big\} \Big\|_p \\
\le\ & 2\Big\| \int_\delta^1 du\, |Y_u(0,u-\delta)|^2 \Big\|_p + 2\Big\| \int_\delta^1 du\, |Y_u(0,u-\delta)|^2 \Big\|_p^{1/2} \Big\| \int_\delta^1 du\, |Y_u(u-\delta,u-\epsilon)|^2 \Big\|_p^{1/2}.
\end{aligned}
$$

But, $\Big\| \int_\delta^1 du\, |Y_u(0,u-\delta)|^2 \Big\|_p \le c_p \int_\delta^1 du(1+\log(1/\delta)) \le c_p \log 1/\delta \le c' p \log\log(1/\epsilon)$, and similarly $\Big\| \int_\delta^1 du\, |Y_u(u-\delta,u-\epsilon)|^2 \Big\|_p \le c_p \int_\delta^1 du(1+\log\delta/\epsilon) \le c_p' \log(1/\epsilon)$. Hence (4.16) is obtained.

To bound the L^p norm of $\int_\delta^1 du \Big\{ |Y_u(u-\delta,u-\epsilon)|^2 \Big\}$, we cut the interval $[\delta, 1]$ into $n = 1/\delta$ parts:

$$
\int_\delta^1 du \Big\{ |Y_u(u-\delta,u-\epsilon)|^2 \Big\} = \sum_{j\le n-1:j \text{ odd}} \int_{j\delta}^{(j+1)\delta} + \sum_{j\le n-1:j \text{ even}} \int_{j\delta}^{(j+1)\delta}.
$$

Remark that the sum on odd $j \le n-1$ is a sum of iid variables whose common law is that of

$$
\int_\delta^{2\delta} \Big\{ |Y_u(u-\delta,u-\epsilon)|^2 \Big\} \overset{\text{law}}{=} \delta \int_1^2 du \left\{ \left| \int_{u-1}^{u-\epsilon/\delta} ds \frac{B_u - B_s}{|B_u - B_s|^3} \right|^2 \right\}.
$$

Applying (3.2), we obtain that

$$
\begin{aligned}
\Big\| \sum_{j\le n-1:j \text{ odd}} \int_{j\delta}^{(j+1)\delta} |Y_u(u-\delta,u-\epsilon)|^2 \Big\|_p &\le c_p n^{1/2} \delta \left\| \int_1^2 du \left\{ \left| \int_{u-1}^{u-\epsilon/\delta} ds \frac{B_u - B_s}{|B_u - B_s|^3} \right|^2 \right\} \right\|_p \\
&\le c_p' n^{1/2} \delta \int_1^2 du(1+\log\delta/\epsilon) \\
&\le c_p' \delta^{1/2} \log(1/\epsilon) \\
&= c_p' (\log 1/\epsilon)^{1-\theta/2}.
\end{aligned}
$$

The same holds for the sum on even j. Since $\theta > 2$, we get $\Big\| \int_\delta^1 du \Big\{ |Y_u(u-\delta,u-\epsilon)|^2 \Big\} \Big\|_p = O(1)$ which in view of (4.15) and (4.16) implies (4.14), as desired. □

5 Proof of Theorem 1.1

Denote by

$$
N_t = -\frac{1}{2\pi} \int_1^t dB_u \int_0^{u-1} ds \frac{B_u - B_s}{|B_u - B_s|^3}, \qquad t > 1.
$$

We claim that

$$
X_t \overset{\text{def}}{=} \big\{ m(S^K(0,t)) \big\} - \mathcal{C}(K)^2 N_t = o\left(\sqrt{t}\, (\log t)^\delta \right), \qquad \text{a.s., } t \to \infty. \tag{5.1}
$$

In fact, by Proposition 3.1, (4.3) and (4.4), we have

$$\|X_t\|_p \leq c_p\sqrt{t}. \tag{5.2}$$

On the other hand, applying (4.5) to N_t, we deduce from Lemma 4.4 and Corollary 3.4 that

$$\begin{aligned} \|X_t - X_s\|_p &\leq c_p\sqrt{(t-s)\log t}, \qquad t > s > 27, \\ \Big\| \sup_{t\leq u\leq t+1} |X_u - X_t| \Big\|_p &\leq c_p\sqrt{\log t}, \qquad t > 27, \end{aligned}$$

which together with (5.2) allows us to apply Proposition 2.2, and shows (5.1).

We can also apply Proposition 2.3 in view of Lemma 4.4, and we obtain that

$$\int_1^t du \left| \int_0^{u-1} ds \frac{B_u - B_s}{|B_u - B_s|^3} \right|^2 - 2t\log t \;=\; o\left(t(\log t)^{\frac{1}{2}+\delta}\right), \qquad \text{a.s., } t \to \infty. \tag{5.3}$$

By Dubins-Schwarz's representation, there exists some one-dimensional Brownian motion β such that

$$\begin{aligned} N_t &= \frac{1}{\pi\sqrt{2}}\beta\left(\frac{1}{2}\int_1^t du \left| \int_0^{u-1} ds \frac{B_u - B_s}{|B_u - B_s|^3} \right|^2\right) \\ &= \frac{1}{\pi\sqrt{2}}\beta(t\log t) + o\left(t^{1/2}(\log t)^{\frac{1}{4}+\delta}\right), \qquad \text{a.s., } t \to \infty, \end{aligned} \tag{5.4}$$

by using (5.3) and the Brownian increments (cf. [4], Theorem 1.2.1): For a non-decreasing function $0 < a_t \leq t$ such that $t/a_t \uparrow$, we have

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2a_t(\log(t/a_t) + \log\log t)}} \sup_{0\leq s\leq t-a_t} \sup_{0\leq v\leq a_t} |\widetilde{\beta}(s+v) - \widetilde{\beta}(s)| = 1, \qquad \text{a.s.}$$

Theorem 1.1 follows by assembling (5.1) and (5.4). □

Acknowledgements. Cooperation between the authors was supported by the joint French-Hungarian Intergovernmental Grant "Balaton" no. F-39/00.

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